无穷区间上迭代泛函分数阶q-差分方程的正解

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摘　要：为了拓展分数阶$q-$差分方程的理论，研究了一类迭代泛函高阶分数阶$q-$差分方程边值问题解的存在性，且这类方程在无穷区间上具有积分边界条件。首先，计算该边值问题的格林函数并分析其特性；其次，给定合适的 Banach 空间以及范数，并据此构造了积分算子；再次，通过运用单调迭代技巧以及上下解方法，成功获得了该问题正解的存在性定理；最后，通过具体的实例验证了主要结果的有效性与实用性。结果表明，在非线性项 g 满足一定的增长条件下，无穷区间上迭代泛函分数阶$q-$差分方程边值问题至少存在一个正解。研究结果丰富了分数阶$q-$差分方程可解性理论，反映了反馈迭代项对解存在性的影响，并为迭代泛函分数阶$q-$差分方程在控制工程、生物医学、人口理论、社会经济等诸多领域中的应用提供了一定的理论指导。

关键词：非线性泛函分析；上下解；单调迭代技术；无穷区间；积分边界条件

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Positive solution of iterative functional fractional q-differential equation on infinite interval

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Abstract: In order to expand the theory of fractional difference equations, the existence of solutions to boundary value problems of a class of iterative functional higher-order fractional q- difference equations with integral boundary conditions on infinite intervals is studied. Firstly, the Green function of fractional difference equation boundary value problem is calculated and its properties are studied. Secondly, given the appropriate Banach space and norm, and the integral operator is constructed. Thirdly, the existence theorem of positive solution of the problem is obtained by using monotone iteration technique and upper and lower solution method. The effectiveness and practicability of the main results are verified by a concrete example. The results show that, under certain growth conditions of the nonlinear term, the iterative functional fractional quantum difference equation boundary value problem has at least one solution. The research results enrich the solvability theory of q-fractional difference equations, reflect the influence of feedback iterative terms on the existence of solutions, and provide certain theoretical guidance for the application of iterative functional fractional q-difference equations in many fields such as control engineering, biomedicine, population theory, and social economy.

Keywords: nonlinear functional analysis; upper and lower solution; monotone iterative technique; infinite intervals; integral boundary conditions

量子微积分，亦称为$q-$微积分，20 世纪初由 JACKSON[1]提出，广泛应用于数学、量子理论和力学等领域[2-5]。20 世纪中叶，Al-Salam[6]和 Agarwal[7]发展了分数阶$q-$微积分的一般性理论。分数阶$q-$微积分在许多领域有着广泛的应用，如宇宙弦与黑洞、航空航天动力学、生物学、经济学、控制论、医学、电学、信号

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处理、图像处理、生物物理学、血流现象等。近年来，分数阶 $q-$ 差分方程可解性理论取得了丰富的研究成果[8-13]。

自反馈是控制理论和工程技术中普遍存在的现象，迭代泛函微分方程是研究自反馈现象的关键工具。作为一种具有复杂偏差变元的方程，其独特之处在于时滞不仅与时间相关，还与状态及其导数密切相关。这种特性使它在描述动态系统中的历史依赖性和非线性行为方面具有重要作用。1984 年 Eder[14]研究了迭代泛函微分方程 $x'(t)=x(x(t))$，引入了区间上饱和解的概念，并运用压缩映像原理得到了解析解的存在性。1993 年 Fečkan[15]利用不动点定理研究了一类迭代泛函微分方程 $u'(x)=f(u(u(x)))$ 解的存在性。这类方程在经典电动力学、力学模型、传染病传播、人口模型、经济系统动力学等领域广泛应用。同时，迭代泛函分数阶微分方程实际应用于数学、物理和工程等领域，关于它的研究涉及到分数阶导数的定义、迭代方法的应用、解析解和数值解的求解以及解的存在唯一性等问题。近年来，迭代泛函分数阶微分方程可解性的研究取得了显著的进展[16-19]。

2013 年，Wang[20]等人研究了参数为 λ 的迭代泛函分数阶微分方程

$$\begin{cases} {}^{C}D_{a,t}^{\alpha}x(t)=f(t,x(t),x(x^{v}(t)))+\lambda, & t\in[a,b],\\ x(t)=\varphi(t),t\in[a_1,a], x(t)=\psi(t), & t\in[b,b_1], \end{cases}$$

解的存在性，其中，${}^{C}D_{a,t}^{\alpha}$ 是 Caputo 分数阶导数，$v\in R\setminus\{0\},\alpha\in(0,1),\lambda\in R$，$a_1\le a<b\le b_1$，$a_1\le a_1^{v}$ 且 $b_1^{v}\le b_1$，$f\in C([a,b]\times[a_1,b_1]^2,R)$；且 $\psi\in C([b,b_1],[a_1,b_1])$。通过 Schauder 不动点定理建立了解的存在性定理。

2022 年，曹[21]等人研究了一类非瞬时脉冲分数阶迭代微分方程

$$\begin{cases} D_{s_i^{+}}^{\alpha}u(t)=f(t,u(t),u(u(t))) \quad t\in(s_i,t_{i+1}],i=0,1,...,m,\\ u(t)=g_i(t),t\in(t_i,s_i],i=1,2,...,m,\\ \Delta u(s_i)=0,\Delta u(t_i)=Q_i(t_i,u(t_i)),i=1,2,...,m,\\ u(0)=0,u(1)=u_0 \end{cases}$$

正解的存在性和唯一性，其中，$D_{s_i^{+}}^{\alpha}u(t)$ 为 Caputo 型分数阶导数，$1<\alpha<2$，$J=[0,1]$，$J_0=J\setminus\{t_1,t_2,...,t_m\}$，$0=s_0<t_1<s_1<\cdots<s_i<t_{i+1}<\cdots<t_{m+1}=1$，$f\in C(J^3,\mathbb{R})$，$Q_i\in C(J^2,\mathbb{R}^{-})$，$i=1,2,\cdots,m$，$\Delta u(s_i)=u(s_i^{+})-u(s_i^{-})$，$\Delta u(t_i)=u(t_i^{+})-u(t_i^{-})$，$u(s_i^{+})$，$u(s_i^{-})$ 是 $u(t)$ 在 s_i 的右极限和左极限，$u(t_i^{+})$，$u(t_i^{-})$ 分别是 $u(t)$ 在 t_i 的右极限和左极限。通过运用 Schauder 不动点定理和 Banach 压缩映射原理分别得到了解的存在性和唯一性。

2023 年，Liu[22]研究了一类无穷区间上迭代泛函分数阶微分方程边值问题（BVP）

$$\begin{cases} {}^{C}D_{0^{+}}^{\alpha}x(t)=f(t,x(t),x'(t),x(x(t))), \quad t\in\mathbb{R}_{+}:=[0,+\infty],\\ x'(0)=a,x''(0)=0,x(\xi)=\int_0^{+\infty}p(t)x(t)dt \end{cases}$$

解的存在性，其中，$2<\alpha<3$，${}^{c}D_{0^{+}}^{\alpha}x(t)=\dfrac{1}{\Gamma(3-\alpha)}\int_0^{t}\dfrac{x''(s)}{(t-s)^{\alpha-2}}ds$ 代表 Caputo 分数阶导数，$f\in C(\mathbb{R}_{+}^{4},\mathbb{R}_{+})$，$p\in L(\mathbb{R}_{+},\mathbb{R}_{+})$，$a,\xi\in\mathbb{R}_{+}$。通过求解微分不等式来确定上述方程的上解，并利用单调迭代技巧得到了解的存在性结果。

迭代泛函分数阶 $q-$ 差分方程在众多领域中有重要应用。例如：复杂系统建模中，它描述记忆性、遗传特性等非局部动力学；金融工程中，刻画高频数据记忆效应与离散跳跃，优化期权定价及风险管理；生物医学中，模拟神经元时滞效应及药物代谢非指数衰减，为跨学科研究提供有力工具等。目前，迭代泛函分数阶微分方程解的存在唯一性已受到众多学者运用不动点定理、单调迭代技巧等方法的广泛研究。但迭代泛函分数阶 $q-$ 差分方程可解性的研究尚未见报道。基于上述研究现状，本文研究一类无穷区间上迭代泛函分数阶 $q-$ 差分方程 BVP

$$\begin{cases} {}^{c}D_{q}^{\alpha}v(t)-g(t,v(t),D_qv(t),v(v(t)))=0, \ t\in J:=[0,+\infty],\\ D_qv(0)=\lambda, \ D_q^2v(0)=D_q^3v(0)=\cdots=D_q^{n-1}v(0)=0,\\ v(\eta)=\int_0^{+\infty}w(t)v(t)d_qt \end{cases} \tag{1}$$

的可解性，其中，$n-1<\alpha\le n$，$n\in\mathbb{N}$ 且 $n\ge3$，λ，$\eta\in J$，$g\in C(J\times J^3,J)$，$w(t)\in L_q^1(J,J)$，${}^{c}D_q^{\alpha}$ 是 Caputo 型分数阶 $q-$ 导数。

本文假设条件如下：

(C_1) 对于 $\forall t \in J$，函数 $g \in C(J \times J^3, J)$ 有

$$0 \le g(t, x_2, y_2, z_2) - g(t, x_1, y_1, z_1)$$

其中 $x_1 \le x_2$，$y_1 \le y_2$，$z_1 \le z_2$；

(C_2) 存在函数 $l_i(t) \in L_q(J, J)$，$i = 1,2,3$，对于 $\forall t, x, y, z \in J$ 有

$$0 \le g(t, x, y, z) \le l_1(t)x + l_2(t)y + l_3(t)z \text{；}$$

$$\int_0^{+\infty} l_1(s)(1 + s^{\alpha-1}) d_q s = l_1 < +\infty \text{，} \int_0^{+\infty} l_2(s)(1 + s^{\alpha-2}) d_q s = l_2 < +\infty \text{，}$$

$$\int_0^{+\infty} l_3(s) d_q s = l_3 < +\infty \text{，} \int_0^{+\infty} l_3(s)(1 + s^{\alpha-1})^{\alpha-1} d_q s = l_4 < +\infty \text{；}$$

(C_3) 函数 $w(t) \in C(J, J)$，

$$0 \le \int_0^{+\infty} w(\tau) d_q \tau < 1 \text{，} \quad 0 \le \int_0^{+\infty} \tau^{\alpha-1} w(\tau) d_q \tau < +\infty \text{，}$$

$$\eta^{\alpha-1} \le \int_\eta^{+\infty} (\tau - \eta)^{(\alpha-1)} w(\tau) d_q \tau < +\infty \text{，} \quad \eta \le \int_0^{+\infty} \tau w(\tau) d_q \tau < +\infty \text{。}$$

为了方便，记 $Q(s) = \int_{qs}^{+\infty} (\tau - qs)^{(\alpha-1)} w(\tau) d_q \tau$，$s \in J$。

1 预备知识

定义 1 若存在函数 $v_0 = v_0(t)$ 且 $\int_0^{+\infty} Q(s)\, {}^c D_q^{\alpha} v_0(s) d_q s < +\infty$，满足

$$\begin{cases} {}^c D_q^{\alpha} v_0(t) \ge g(t, v_0(t), D_q v_0(t), v_0(v_0(t))), \ t \in J, \\ D_q v_0(0) \ge \lambda, \ D_q^2 v_0(0) = D_q^3 v_0(0) = \cdots = D_q^{n-1} v_0(0) = 0, \\ v_0(\eta) \ge \int_0^{+\infty} w(t) v_0(t) d_q t, \end{cases} \tag{2}$$

则称 $v_0 = v_0(t)$ 是 BVP（1）的上解。

引理 1[23] 设 $\alpha, \beta \ge 0$，函数 f 在[0, 1]上连续且导数存在，则下列等式成立：

(i) $(D_q^{\alpha} I_q^{\alpha} f)(t) = f(t)$；

(ii) $(I_q^{\alpha} I_q^{\beta} f)(t) = (I_q^{\alpha+\beta} f)(t)$。

引理 2[24-25] 设 $\alpha > 0$ 且 n=[α]+1，则

$$(I_q^{\alpha\, c} D_q^{\alpha} f)(t) = f(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1} \text{，}$$

其中 $c_0, c_1, \cdots, c_{n-1}$ 是任意常数。

引理 3 若 (C_3) 成立, 函数 $y(t) \in C(J, J)$，$\int_0^{+\infty} Q(s) y(s) d_q s < +\infty$，则分数阶 $q-$差分方程 BVP

$$\begin{cases} {}^c D_q^{\alpha} v(t) = y(t), \ t \in J, \\ D_q v(0) = \lambda, \ D_q^2 v(0) = D_q^3 v(0) = \cdots = D_q^{n-1} v(0) = 0, \\ v(\eta) = \int_0^{+\infty} w(t) v(t) d_q t, \end{cases} \tag{3}$$

有唯一解

$$v(t) = \int_0^{+\infty} U(t, qs) y(s) d_q s + \psi(t) \tag{4}$$

且

$$D_q v(t) = \frac{1}{\Gamma_q(\alpha - 1)} \int_0^t (t - qs)^{(\alpha-2)} y(s) d_q s + \lambda \text{，}$$

其中

$$U(t,qs)=\frac{1}{\Lambda\Gamma_q(\alpha)}\begin{cases}\int_{qs}^{+\infty}(\tau-qs)^{(\alpha-1)}w(\tau)d_q\tau+\Lambda(t-qs)^{(\alpha-1)}-(\eta-qs)^{(\alpha-1)},0\le qs\le min\{\eta,t\}<+\infty,\\ \int_{qs}^{+\infty}(\tau-qs)^{(\alpha-1)}w(\tau)d_q\tau-(\eta-qs)^{(\alpha-1)},0\le t\le qs<+\infty,\\ \int_{qs}^{+\infty}(\tau-qs)^{(\alpha-1)}w(\tau)d_q\tau+\Lambda(t-qs)^{(\alpha-1)},\eta\le qs\le t<+\infty,\\ \int_{qs}^{+\infty}(\tau-qs)^{(\alpha-1)}w(\tau)d_q\tau,max\{\eta,t\}\le qs<+\infty,\end{cases}\tag{5}$$

$$\psi(t)=\frac{\lambda}{\Lambda}(\int_0^{+\infty}\tau w(\tau)d_q\tau-\eta)+\lambda t\ ,\qquad \Lambda=1-\int_0^{+\infty}w(\tau)d_q\tau\ 。$$

证明 首先，对方程（3）两边进行 α 阶 q-积分，则有

$$v(t)=c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}+I_q^{\alpha}y(t)\ ,$$

由边界条件 $D_qv(0)=\lambda$ 和 $D_q^2v(0)=D_q^3v(0)=\cdots=D_q^{n-1}v(0)=0$，可得 $c_1=\lambda$ 且 $c_2=c_3=\cdots c_{n-1}=0$。因此，

$$v(t)=c_0+\lambda t+\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}y(s)d_qs\ ,$$

又由 $v(\eta)=\int_0^{+\infty}w(t)v(t)d_qt$，可得

$$c_0=\frac{1}{\Lambda\Gamma_q(\alpha)}\left[\int_0^{+\infty}\int_{qs}^{+\infty}w(\tau)(\tau-qs)^{(\alpha-1)}d_q\tau y(s)d_qs-\int_0^{\eta}(\eta-qs)^{(\alpha-1)}y(s)d_qs\right]+\frac{\lambda}{\Lambda}\left(\int_0^{+\infty}\tau w(\tau)d_q\tau-\eta\right)。$$

于是，

$$\begin{aligned}v(t)&=\frac{1}{\Lambda\Gamma_q(\alpha)}\left[\int_0^{+\infty}\int_{qs}^{+\infty}w(\tau)(\tau-qs)^{(\alpha-1)}d_q\tau y(s)d_qs-\int_0^{\eta}(\eta-qs)^{(\alpha-1)}y(s)d_qs+\Lambda\int_0^t(t-qs)^{(\alpha-1)}y(s)d_qs\right]\\&\quad+\frac{\lambda}{\Lambda}\left(\int_0^{+\infty}\tau w(\tau)d_q\tau-\eta\right)+\lambda t\\&=\frac{1}{\Lambda\Gamma_q(\alpha)}\left[\int_0^{+\infty}Q(s)y(s)d_qs-\int_0^{\eta}(\eta-qs)^{(\alpha-1)}y(s)d_qs+\Lambda\int_0^t(t-qs)^{(\alpha-1)}y(s)d_qs\right]+\psi(t)\\&=\int_0^{+\infty}U(t,qs)y(s)d_qs+\psi(t)\ ,\end{aligned}$$

且

$$\begin{aligned}D_qv(t)&=D_q(c_0+\lambda t+\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}y(s)d_qs)\\&=\frac{1}{\Gamma_q(\alpha-1)}\int_0^t(t-qs)^{(\alpha-2)}y(s)d_qs+\lambda\ ,\end{aligned}$$

证毕。

引理 4 假设 $(C_1)-(C_3)$ 成立，则对于 $\forall(t,qs)\in J\times J$，函数 $U(t,qs)$ 和 $\psi(t)$ 具有下列性质：

（1）函数 $U(t,qs)$ 是连续的；

（2）$0\le U(t,qs)\le\frac{1}{\Lambda\Gamma_q(\alpha)}\left(\int_0^{+\infty}\tau^{\alpha-1}w(\tau)d_q\tau+\Lambda t^{\alpha-1}\right)$；

（3）$\psi(t)\ge0$。

证明 由条件 (C_3) 可知（1）和（3）显然成立，下面只需证明（2）成立。

显然 $\Lambda>0$，当 $\eta\le qs<+\infty$ 时，$U(t,qs)\ge0$ 成立；

当 $0\le qs\le\eta$ 时，$(\eta-qs)^{(\alpha-1)}\le\eta^{\alpha-1}\le\int_{\eta}^{+\infty}(\tau-\eta)^{(\alpha-1)}w(\tau)d_q\tau\le\int_{qs}^{+\infty}(\tau-qs)^{(\alpha-1)}w(\tau)d_q\tau$，

故 $0\le qs\le\eta$ 时，$U(t,qs)\ge0$。于是，$\forall(t,qs)\in J\times J$，$U(t,qs)\ge0$。

另一方面，对于 $\forall(t,qs)\in J\times J$，可得

$$U(t,qs)\le\frac{1}{\Lambda\Gamma_q(\alpha)}\left(\int_{qs}^{+\infty}(\tau-qs)^{(\alpha-1)}w(\tau)d_q\tau+\Lambda(t-qs)^{(\alpha-1)}\right)\le\frac{1}{\Lambda\Gamma_q(\alpha)}\left(\int_0^{+\infty}\tau^{\alpha-1}w(\tau)d_q\tau+\Lambda t^{\alpha-1}\right)\ ;$$

证毕。

定义空间 E：

$$E=\left\{v\in C_q^1(J,J)\,|\,\left\|\frac{v(t)}{1+t^{\alpha-1}}\right\|_\infty<+\infty,\left\|\frac{D_qv(t)}{1+t^{\alpha-2}}\right\|_\infty<+\infty\right\},$$

且赋予范数 $\|v\|_E=\max\left\{\left\|\frac{v(t)}{1+t^{\alpha-1}}\right\|_\infty,\left\|\frac{D_qv(t)}{1+t^{\alpha-2}}\right\|_\infty\right\}$，其中 $\|v(t)\|_\infty=\sup_{t\in J}|v(t)|$。易证 $(E,\|\cdot\|_E)$ 为 Banach 空间。值得注意的是，Arzela-Ascoli 定理在无穷区间上是失效的，因此需要引入以下紧性判定准则。

引理 5 设 $X\subset E$ 是一个有界子集。若满足

(i) $\left\{\frac{v(t)}{1+t^{\alpha-1}}\middle|v\in X\right\}$，$\left\{\frac{D_qv(t)}{1+t^{\alpha-2}}\middle|v\in X\right\}$ 在 J 的任意紧子区间上是等度连续的；

(ii) $\forall\varepsilon>0$，$\exists\Upsilon=\Upsilon(\varepsilon)>0$，使得 $\forall t_1,t_2>\Upsilon(\varepsilon)$，$v\in X$，有

$$\left|\frac{v(t_2)}{1+t_2^{\alpha-1}}-\frac{v(t_1)}{1+t_1^{\alpha-1}}\right|<\varepsilon,\quad\left|\frac{D_qv(t_2)}{1+t_2^{\alpha-2}}-\frac{D_qv(t_1)}{1+t_1^{\alpha-2}}\right|<\varepsilon$$

成立，则 X 在 E 中是相对紧的。

证明 方法类似于文献[26]中的引理 2.3 的证明。

引理 6 若 (C_2) 成立，$\forall v\in E$，有

$$\int_0^{+\infty}|g(t,v(t),D_qv(t),v(v(t)))|\,d_qt\le\sum_{i=1}^{3}l_i\|v\|+l_4\|v\|^\alpha \tag{6}$$

证明 $\forall v\in E$，由 (C_2) 知

$$\begin{aligned}
&\int_0^{+\infty}|g(t,v(t),D_qv(t),v(v(t)))|\,d_qt\\
&\le\int_0^{+\infty}|l_1(t)v(t)+l_2(t)D_qv(t)+l_3(t)v(v(t))|\,d_qt\\
&=\int_0^{+\infty}|l_1(t)(1+t^{\alpha-1})\frac{v(t)}{1+t^{\alpha-1}}+l_2(t)(1+t^{\alpha-2})\frac{D_qv(t)}{1+t^{\alpha-2}}+l_3(t)(1+v^{\alpha-1}(t))\frac{v(t)}{1+v^{\alpha-1}(t)}|\,d_qt\\
&\le\int_0^{+\infty}|l_1(t)(1+t^{\alpha-1})+l_2(t)(1+t^{\alpha-2})+l_3(t)(1+v^{\alpha-1}(t))|\,d_qt\cdot\|v\|\\
&\le\int_0^{+\infty}|l_1(t)(1+t^{\alpha-1})+l_2(t)(1+t^{\alpha-2})+l_3(t)(1+(1+t^{\alpha-1})^{\alpha-1})|\,d_qt\cdot\|v\|\\
&=\left[\int_0^{+\infty}l_1(t)(1+t^{\alpha-1})+\int_0^{+\infty}l_2(t)(1+t^{\alpha-2})+\int_0^{+\infty}l_3(t)d_qt\right]\|v\|+\int_0^{+\infty}l_3(t)(1+(1+t^{\alpha-1})^{\alpha-1})d_qt\cdot\|v\|^\alpha\\
&=(l_1+l_2+l_3)\|v\|+l_4\|v\|^\alpha\\
&=\sum_{i=1}^{3}l_i\|v\|+l_4\|v\|^\alpha。
\end{aligned}$$

证毕。

2 主要结论

定义算子 T：$P\to E$ 为

$$(Tv)(t)=\int_0^{+\infty}U(t,qs)g(s,v(s),D_qv(s),v(v(s)))d_qs+\psi(t),\qquad t\in J,$$

且

$$D_q(Tv)(t)=\frac{1}{\Gamma_q(\alpha-1)}\int_0^t(t-qs)^{(\alpha-2)}g(s,v(s),D_qv(s),v(v(s)))d_qs+\lambda,$$

其中 $P=\left\{v\in E\middle|v(t)\ge0,D_qv(t)\ge0,t\in J\right\}$ 是一个锥。

显然，$v\in P$ 是方程（1）的正解当且仅当 v 是算子 T 的不动点。

引理 7 假设 (C_1)-(C_3) 成立，则算子 T：$P\to P$ 是全连续的。

证明 首先，证明 $\forall v\in P$ 使得 $Tv\in P$。由引理 4 和 (C_1) 易得 $(Tv)(t)\ge0$，$D_q(Tv)(t)\ge0$。

应用引理 4 和引理 6，可得

$$0\le\frac{Tv(t)}{1+t^{\alpha-1}}=\int_0^{+\infty}\frac{U(t,qs)}{1+t^{\alpha-1}}g(s,v(s),D_qv(s),v(v(s)))d_qs+\frac{\psi(t)}{1+t^{\alpha-1}}$$

$$\le \int_0^{+\infty} \frac{\frac{1}{\Lambda\Gamma_q(\alpha)}\left(\int_0^{+\infty}\tau^{\alpha-1}w(\tau)d_q\tau+\Lambda t^{\alpha-1}\right)}{1+t^{\alpha-1}} g(s,v(s),D_qv(s),v(v(s)))d_qs+\frac{\psi(t)}{1+t^{\alpha-1}}$$

$$\le \int_0^{+\infty} \frac{K+\Lambda t^{\alpha-1}}{\Lambda\Gamma_q(\alpha)(1+t^{\alpha-1})} g(s,v(s),D_qv(s),v(v(s)))d_qs+\frac{\psi(t)}{1+t^{\alpha-1}}$$

$$\le \left(\sum_{i=1}^{3} l_i \| v\| + l_4 \| v\|^{\alpha}\right)\frac{K+\Lambda t^{\alpha-1}}{\Lambda\Gamma_q(\alpha)(1+t^{\alpha-1})}+\frac{\psi(t)}{1+t^{\alpha-1}}$$

$$\le \frac{K+\Lambda}{\Lambda\Gamma_q(\alpha)}\left(\sum_{i=1}^{3} l_i \| v\| + l_4 \| v\|^{\alpha}\right)+\frac{\psi(t)}{1+t^{\alpha-1}}$$

$$< +\infty,$$

且

$$0 \le \frac{D_q(Tv)(t)}{1+t^{\alpha-2}} = \frac{1}{\Gamma_q(\alpha-1)}\int_0^t \frac{(t-qs)^{(\alpha-2)}}{1+t^{\alpha-2}} g(s,v(s),D_qv(s),v(v(s)))d_qs+\frac{\lambda}{1+t^{\alpha-2}}$$

$$\le \frac{1}{\Gamma_q(\alpha-1)}\frac{t^{\alpha-2}}{1+t^{\alpha-2}}\int_0^t g(s,v(s),D_qv(s),v(v(s)))d_qs+\lambda$$

$$\le \frac{1}{\Gamma_q(\alpha-1)}\int_0^t g(s,v(s),D_qv(s),v(v(s)))d_qs+\lambda$$

$$\le \frac{1}{\Gamma_q(\alpha-1)}\int_0^{+\infty} g(s,v(s),D_qv(s),v(v(s)))d_qs+\lambda$$

$$\le \frac{1}{\Gamma_q(\alpha-1)}\left(\sum_{i=1}^{3} l_i \| v\| + l_4 \| v\|^{\alpha}\right)+\lambda$$

$$< +\infty,$$

其中，$K=\int_0^{+\infty}\tau^{\alpha-1}w(\tau)d_q\tau$。因此，$Tv\in E$，$(Tv)(t)\ge 0$，$D_q(Tv)(t)\ge 0$，可得$\forall v\in P$，$Tv\in P$，即$T$：$P\to P$。

其次，证明 T：$P\to P$是连续的。设v_n，$v\in P$且$n\to\infty$时$v_n\to v$，根据函数g的连续性及Lebesgue控制收敛定理，有

$$\left\|\frac{Tv_n}{1+t^{\alpha-1}}-\frac{Tv}{1+t^{\alpha-1}}\right\|_{\infty} = \sup_{t\in R_+}\left|\int_0^{+\infty}\frac{U(t,qs)}{1+t^{\alpha-1}}\left(g(s,v_n(s),D_qv_n(s),v_n(v_n(s)))-g(s,v(s),D_qv(s),v(v(s)))\right)d_qs\right|$$

$$\le \frac{K+\Lambda}{\Lambda\Gamma_q(\alpha)}\int_0^{+\infty}|\,g(s,v_n(s),D_qv_n(s),v_n(v_n(s)))-g(s,v(s),D_qv(s),v(v(s)))\,|\,d_qs$$

$$\to 0\quad (n\to\infty)\quad,$$

且

$$\left\|\frac{D_q(Tv_n)}{1+t^{\alpha-2}}-\frac{D_q(Tv)}{1+t^{\alpha-2}}\right\|_{\infty}$$

$$= \sup_{t\in R_+}\left|\frac{1}{\Gamma_q(\alpha-1)}\int_0^t\frac{(t-qs)^{(\alpha-2)}}{1+t^{\alpha-2}}\left(g(s,v_n(s),D_qv_n(s),v_n(v_n(s)))-g(s,v(s),D_qv(s),v(v(s)))\right)d_qs\right|$$

$$\le \frac{t^{\alpha-2}}{\Gamma_q(\alpha-1)(1+t^{\alpha-2})}\int_0^{+\infty}|g(s,v_n(s),D_qv_n(s),v_n(v_n(s)))-g(s,v(s),D_qv(s),v(v(s)))\,|\,d_qs$$

$$\le \frac{1}{\Gamma_q(\alpha-1)}\int_0^{+\infty}|g(s,v_n(s),D_qv_n(s),v_n(v_n(s)))-g(s,v(s),D_qv(s),v(v(s)))\,|\,d_qs$$

$$\to 0\quad (n\to\infty),$$

综上，$\|Tv_n-Tv\|\to 0\quad (n\to\infty)$。故算子$T$：$P\to P$是连续的。

再次，证明算子T是一致有界的。定义有界集$\Omega=\{v(t)\in P:\|v(t)\|_E\le M\}$，对于任意的$v(t)\in\Omega$，由引理4和引理6，有

$$\left\|\frac{Tv}{1+t^{\alpha-1}}\right\|_\infty = \sup_{t\in R_+}\left|\int_0^{+\infty}\frac{U(t,qs)}{1+t^{\alpha-1}}g(s,v_n(s),D_qv_n(s),v_n(v_n(s)))d_qs\right| + \frac{\psi(t)}{1+t^{\alpha-1}}$$

$$\le \frac{K+\Lambda}{\Lambda\Gamma_q(\alpha)}\left(\sum_{i=1}^{3}l_i\|v\|+l_4\|v\|^\alpha\right)+\frac{\frac{\lambda}{\Lambda}(\int_0^{+\infty}\tau w(\tau)d_q\tau-\eta)+\lambda t}{1+t^{\alpha-1}}$$

$$\le \frac{K+\Lambda}{\Lambda\Gamma_q(\alpha)}\left(\sum_{i=1}^{3}l_i\|v\|+l_4\|v\|^\alpha\right)+\frac{\lambda}{\Lambda}(c-\eta)+\lambda,$$

且

$$\left\|\frac{D_q(Tv)(t)}{1+t^{\alpha-2}}\right\|_\infty = \sup_{t\in R_+}\left|\frac{1}{\Gamma_q(\alpha-1)}\int_0^t\frac{(t-qs)^{(\alpha-2)}}{1+t^{\alpha-2}}g(s,v(s),D_qv(s),v(v(s)))d_qs\right|+\frac{\lambda}{1+t^{\alpha-2}}$$

$$\le \frac{1}{\Gamma_q(\alpha-1)}\int_0^t g(s,v(s),D_qv(s),v(v(s)))d_qs+\lambda$$

$$\le \frac{1}{\Gamma_q(\alpha-1)}\int_0^{+\infty} g(s,v(s),D_qv(s),v(v(s)))d_qs+\lambda$$

$$\le \frac{1}{\Gamma_q(\alpha-1)}\left(\sum_{i=1}^{3}l_i\|v\|+l_4\|v\|^\alpha\right)+\lambda\ ,$$

其中， $c=\int_0^{+\infty}\tau w(\tau)d_q\tau$ 。因此，

$$\|Tv\|\le\max\left\{\frac{K+\Lambda}{\Lambda\Gamma_q(\alpha)}\left(\sum_{i=1}^{3}l_i\|v\|+l_4\|v\|^\alpha\right)+\frac{\lambda}{\Lambda}(c-\eta)+\lambda;\right\},$$

即 T 是一致有界的。

然后，证明 T 在 J 的任意紧子区间 J_1 上是等度连续的。对于任意的 $v(t)\in\Omega$，当 $t_1,t_2\in J_1$ 且 $t_2>t_1$ 时，有

$$\left|\frac{(Tv)(t_2)}{1+t_2^{\alpha-1}}-\frac{(Tv)(t_1)}{1+t_1^{\alpha-1}}\right|$$

$$\le\frac{1}{\Lambda\Gamma_q(\alpha)}\left|\frac{1}{1+t_2^{\alpha-1}}-\frac{1}{1+t_1^{\alpha-1}}\right|\left[\int_0^{+\infty}Q(s)G_v(s)d_qs-\int_0^{\eta}(\eta-qs)^{(\alpha-1)}G_v(s)d_qs\right]$$

$$+\frac{1}{\Gamma_q(\alpha)}\int_0^{t_1}\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}-\frac{(t_1-qs)^{(\alpha-1)}}{1+t_1^{\alpha-1}}\right|G_v(s)d_qs+\int_{t_1}^{t_2}\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}\right|G_v(s)d_qs$$

$$+\left|\frac{1}{1+t_2^{\alpha-1}}-\frac{1}{1+t_1^{\alpha-1}}\right|\frac{\lambda}{\Lambda}(c-\eta)+\lambda\left|\frac{t_2}{1+t_2^{\alpha-1}}-\frac{t_1}{1+t_1^{\alpha-1}}\right|$$

$$\le\frac{1}{\Lambda\Gamma_q(\alpha)}\left|\frac{t_1^{\alpha-1}-t_2^{\alpha-1}}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})}\right|\left[\int_0^{+\infty}Q(s)G_v(s)d_qs-\int_0^{\eta}(\eta-qs)^{(\alpha-1)}G_v(s)d_qs\right]$$

$$+\frac{1}{\Gamma_q(\alpha)}\int_0^{t_1}\left|\frac{(t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}\right|G_v(s)d_qs$$

$$+\int_{t_1}^{t_2}\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}\right|G_v(s)d_qs+\left|\frac{t_1^{\alpha-1}-t_2^{\alpha-1}}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})}\right|\frac{\lambda}{\Lambda}(c-\eta)+\lambda\left|\frac{t_2-t_1}{1+t_2^{\alpha-1}}\right|$$

$$\to 0\quad(t_1\to t_2)\ ,$$

且

$$\left|\frac{D_q(Tv)(t_2)}{1+t_2^{\alpha-2}}-\frac{D_q(Tv)(t_1)}{1+t_1^{\alpha-2}}\right|$$

$$=\frac{1}{\Gamma_q(\alpha-1)}\left|\frac{\int_0^{t_2}(t_2-qs)^{(\alpha-2)}G_v(s)d_qs+\lambda}{1+t_2^{\alpha-2}}-\frac{\int_0^{t_1}(t_1-qs)^{(\alpha-2)}G_v(s)d_qs+\lambda}{1+t_1^{\alpha-2}}\right|$$

$$\le \frac{1}{\Gamma_q(\alpha-1)}\left|\int_0^{t_1}[\frac{(t_2-qs)^{(\alpha-2)}}{1+t_2^{\alpha-2}}-\frac{(t_1-qs)^{(\alpha-2)}}{1+t_1^{\alpha-2}}]G_v(s)d_qs-\int_{t_1}^{t_2}\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}\right|G_v(s)d_qs\right|$$

$$+\lambda\left|\frac{1}{1+t_2^{\alpha-2}}-\frac{1}{1+t_1^{\alpha-2}}\right|$$

$$\le \frac{1}{\Gamma_q(\alpha-1)}\int_0^{t_1}\left|\frac{(t_2-qs)^{(\alpha-2)}-(t_1-qs)^{(\alpha-2)}}{1+t_2^{\alpha-2}}\right|G_v(s)d_qs$$

$$+\frac{1}{\Gamma_q(\alpha-1)}\int_{t_1}^{t_2}\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}\right|G_v(s)d_qs+\lambda\left|\frac{t_1^{\alpha-2}-t_2^{\alpha-2}}{(1+t_2^{\alpha-2})(1+t_1^{\alpha-2})}\right|$$

$$\to 0\ \ (t_1\to t_2)\,,$$

其中 $G_v(s)=g(s,v(s),D_qv(s),v(v(s)))$。于是，算子 T 是等度连续的。

最后，证明 T 在无穷远处是等度收敛的。由引理 5 可知 $\forall\varepsilon>0$，$\exists\Upsilon=\Upsilon(\varepsilon)$ 使得 $\forall t_1,t_2>\Upsilon$，有

$$\left|\frac{v(t_2)}{1+t_2^{\alpha-1}}-\frac{v(t_1)}{1+t_1^{\alpha-1}}\right|<\varepsilon\quad,\quad\left|\frac{D_qv(t_2)}{1+t_2^{\alpha-2}}-\frac{D_qv(t_1)}{1+t_1^{\alpha-2}}\right|<\varepsilon\quad,$$

进一步，对于 $\forall\varepsilon>0$，存在常数 $L>0$，使得

$$\int_L^{+\infty}G_v(s)d_qs<\varepsilon\quad。$$

另一方面 $\lim\limits_{t\to\infty}\frac{t^\gamma}{1+t^\gamma}=1$，$\gamma\in\{\alpha-1,\alpha-2\}$，则存在足够大的 $\Upsilon_1>0$，使得对于 $\forall t_2>t_1>\Upsilon_1$，有

$$\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|<\varepsilon\quad,\quad\left|\frac{t_2^{\alpha-2}}{1+t_2^{\alpha-2}}-\frac{t_1^{\alpha-2}}{1+t_1^{\alpha-2}}\right|<\varepsilon\quad。$$

类似地 $\lim\limits_{t\to\infty}\frac{(t-L)^{(\rho-1)}}{1+t^{\rho-1}}=1$，$\rho=\{\alpha-1,\alpha\}$，存在足够大的 $\Upsilon_2>L$ 使得 $\forall t_2>t_1>\Upsilon_2$ 且 $0\le qs\le L$，有

$$\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}-\frac{(t_1-qs)^{(\alpha-1)}}{1+t_1^{\alpha-1}}\right|<\varepsilon,\quad\left|\frac{(t_2-qs)^{(\alpha-2)}}{1+t_2^{\alpha-2}}-\frac{(t_1-qs)^{(\alpha-2)}}{1+t_1^{\alpha-2}}\right|<\varepsilon,$$

同时，因为 $\lim\limits_{t\to\infty}\frac{t}{1+t^\alpha}=0$，那么 $\exists\Upsilon_3>0$，$\forall t_2>t_1>\Upsilon_3$ 使得 $\left|\frac{t_2}{1+t_2^{\alpha-1}}-\frac{t_1}{1+t_1^{\alpha-1}}\right|<\varepsilon$。

取 $\Upsilon=\max\{\Upsilon_1,\Upsilon_2,\Upsilon_3\}$，对于任意的 $v(t)\in\Omega$，$t_2>t_1>\Upsilon$，当 $t_1\to t_2$，有

$$\left|\frac{(Tv)(t_2)}{1+t_2^{\alpha-1}}-\frac{(Tv)(t_1)}{1+t_1^{\alpha-1}}\right|$$

$$\le\frac{1}{\Lambda\Gamma_q(\alpha)}\left|\frac{1}{1+t_2^{\alpha-1}}-\frac{1}{1+t_1^{\alpha-1}}\right|\left[\int_0^{+\infty}Q(s)G_v(s)d_qs-\int_0^{\eta}(\eta-qs)^{(\alpha-1)}G_v(s)d_qs\right]$$

$$+\frac{1}{\Gamma_q(\alpha)}\left(\int_0^{t_1}\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}-\frac{(t_1-qs)^{(\alpha-1)}}{1+t_1^{\alpha-1}}\right|G_v(s)d_qs+\int_{t_1}^{t_2}\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}\right|G_v(s)d_qs\right)$$

$$+\frac{\lambda}{\Lambda}(c-\eta)\left|\frac{1}{1+t_2^{\alpha-1}}-\frac{1}{1+t_1^{\alpha-1}}\right|+\lambda\left|\frac{t_2}{1+t_2^{\alpha-1}}-\frac{t_1}{1+t_1^{\alpha-1}}\right|$$

$$\le\frac{1}{\Lambda\Gamma_q(\alpha)}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\left[\int_0^{+\infty}Q(s)G_v(s)d_qs+\max_{s\in[0,\eta],v\in\Omega}G_v(s)\int_0^{\eta}(\eta-qs)^{(\alpha-1)}d_qs\right]$$

$$+\frac{1}{\Gamma_q(\alpha)}\int_0^{t_1}\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}-\frac{(t_1-qs)^{(\alpha-1)}}{1+t_1^{\alpha-1}}\right|G_v(s)d_qs$$

$$+\frac{1}{\Gamma_q(\alpha)}\int_{t_1}^{t_2}\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}\right|G_v(s)d_qs+\frac{\lambda}{\Lambda}(c-\eta)\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|+\lambda\left|\frac{t_2}{1+t_2^{\alpha-1}}-\frac{t_1}{1+t_1^{\alpha-1}}\right|$$

$$=\frac{1}{\Lambda\Gamma_q(\alpha)}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\int_0^{+\infty}Q(s)G_v(s)d_qs+\frac{\max\limits_{s\in[0,\eta],v\in\Omega}G_v(s)}{\Lambda\Gamma_q(\alpha+1)}\eta^\alpha\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|$$

$$
\begin{aligned}
&+\frac{1}{\Gamma_q(\alpha)}\int_0^{t_1}\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}-\frac{(t_1-qs)^{(\alpha-1)}}{1+t_1^{\alpha-1}}\right|G_v(s)d_qs\\
&+\frac{1}{\Gamma_q(\alpha)}\int_{t_1}^{t_2}\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}\right|G_v(s)d_qs+\frac{\lambda}{\Lambda}(c-\eta)\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|+\lambda\left|\frac{t_2}{1+t_2^{\alpha-1}}-\frac{t_1}{1+t_1^{\alpha-1}}\right|\\
\le&\frac{K}{\Lambda\Gamma_q(\alpha)}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\int_0^{+\infty}G_v(s)d_qs+\frac{\max\limits_{s\in[0,\eta],v\in\Omega}G_v(s)}{\Lambda\Gamma_q(\alpha+1)}\eta^{\alpha}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\\
&+\frac{1}{\Gamma_q(\alpha)}\int_0^{L}\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}-\frac{(t_1-qs)^{(\alpha-1)}}{1+t_1^{\alpha-1}}\right|G_v(s)d_qs\\
&+\frac{1}{\Gamma_q(\alpha)}\int_L^{+\infty}\left[\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}-\frac{(t_1-qs)^{(\alpha-1)}}{1+t_1^{\alpha-1}}\right|+\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}\right|\right]G_v(s)d_qs\\
&+\frac{\lambda}{\Lambda}(c-\eta)\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|+\lambda\left|\frac{t_2}{1+t_2^{\alpha-1}}-\frac{t_1}{1+t_1^{\alpha-1}}\right|\\
\le&\frac{KM^*}{\Lambda\Gamma_q(\alpha)}\varepsilon+\frac{\eta^{\alpha}\max\limits_{s\in[0,\eta],v\in\Omega}G_v(s)}{\Lambda\Gamma_q(\alpha+1)}\varepsilon+\frac{\max\limits_{s\in[0,L],v\in\Omega}G_v(s)}{\Gamma_q(\alpha)}\int_0^{L}\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}-\frac{(t_1-qs)^{(\alpha-1)}}{1+t_1^{\alpha-1}}\right|d_qs\\
&+\frac{3}{\Gamma_q(\alpha)}\int_L^{+\infty}G_v(s)d_qs+\frac{\lambda}{\Lambda}(c-\eta)\varepsilon+\lambda\varepsilon\\
\le&\left[\frac{KM^*}{\Lambda\Gamma_q(\alpha)}+\frac{\eta^{\alpha}\max\limits_{s\in[0,\eta],v\in\Omega}G_v(s)}{\Lambda\Gamma_q(\alpha+1)}+\frac{3+\max\limits_{s\in[0,L],v\in\Omega}G_v(s)}{\Gamma_q(\alpha)}+\frac{\lambda}{\Lambda}(c-\eta)+\lambda\right]\varepsilon,
\end{aligned}
$$

且

$$
\begin{aligned}
&\left|\frac{D_q(Tv)(t_2)}{1+t_2^{\alpha-2}}-\frac{D_q(Tv)(t_1)}{1+t_1^{\alpha-2}}\right|\\
=&\frac{1}{\Gamma_q(\alpha-1)}\left|\frac{\int_0^{t_2}(t_2-qs)^{(\alpha-2)}G_v(s)d_qs+\lambda}{1+t_2^{\alpha-2}}-\frac{\int_0^{t_1}(t_1-qs)^{(\alpha-2)}G_v(s)d_qs+\lambda}{1+t_1^{\alpha-2}}\right|\\
\le&\frac{1}{\Gamma_q(\alpha-1)}\left|\int_0^{t_1}[\frac{(t_2-qs)^{(\alpha-2)}}{1+t_2^{\alpha-2}}-\frac{(t_1-qs)^{(\alpha-2)}}{1+t_1^{\alpha-2}}]G_v(s)d_qs-\int_{t_1}^{t_2}\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}\right|G_v(s)d_qs\right|\\
&+\lambda\left|\frac{1}{1+t_2^{\alpha-2}}-\frac{1}{1+t_1^{\alpha-2}}\right|\\
\le&\frac{1}{\Gamma_q(\alpha-1)}\int_0^{t_1}\left|\frac{(t_2-qs)^{(\alpha-2)}-(t_1-qs)^{(\alpha-2)}}{1+t_2^{\alpha-2}}\right|G_v(s)d_qs\\
&+\frac{1}{\Gamma_q(\alpha-1)}\int_{t_1}^{t_2}\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}\right|G_v(s)d_qs+\lambda\left|\frac{t_1^{\alpha-2}-t_2^{\alpha-2}}{(1+t_2^{\alpha-2})(1+t_1^{\alpha-2})}\right|\\
\le&\frac{1}{\Gamma_q(\alpha-1)}\int_0^{L}\left|\frac{(t_2-qs)^{(\alpha-2)}}{1+t_2^{\alpha-2}}-\frac{(t_1-qs)^{(\alpha-2)}}{1+t_1^{\alpha-2}}\right|G_v(s)d_qs\\
&+\frac{1}{\Gamma_q(\alpha-1)}\int_L^{+\infty}\left(\left|\frac{(t_2-qs)^{(\alpha-2)}}{1+t_2^{\alpha-2}}-\frac{(t_1-qs)^{(\alpha-2)}}{1+t_1^{\alpha-2}}\right|+\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}\right|\right)G_v(s)d_qs+\lambda\varepsilon\\
\le&\frac{\varepsilon L}{\Gamma_q(\alpha-1)}\max_{s\in[0,L],v\in\Omega}G_v(s)+\frac{3}{\Gamma_q(\alpha-1)}\int_L^{+\infty}G_v(s)d_qs+\lambda\varepsilon\\
\le&\left[\frac{L\max\limits_{s\in[0,L],v\in\Omega}G_v(s)+3}{\Gamma_q(\alpha-1)}+\lambda\right]\varepsilon\ 。
\end{aligned}
$$

综上可得，算子T是等度收敛的。于是，T：$P\to P$是全连续的。

引理 8 若 (C_3) 成立， $v\in E$ ，且 $\int_0^{+\infty}Q(s)\,{}^cD_q^{\alpha}v(s)d_qs<+\infty$ ，并满足

$$\begin{cases} {}^cD_q^{\alpha}v(t)\geq y(t),\ t\in J,\\ D_qv(0)\geq 0,\ D_q^2v(0)=D_q^3v(0)=\cdots=D_q^{n-1}v(0)=0,\\ v(\eta)\geq\int_0^{+\infty}w(t)v(t)d_qt, \end{cases}$$

则当 $t\in J$ ， $v\in P$ 时 $v(t)\geq 0$ ， $D_qv(t)\geq 0$ 。

证明 设 ${}^cD_q^{\alpha}v(t)=y(t)\geq 0$, $D_qv(0)=m\geq 0$ ， $d\geq 0$ ， $v(\eta)=\int_0^{+\infty}w(t)v(t)d_qt+d$ ，有

$$\begin{cases} {}^cD_q^{\alpha}v(t)=y(t),\ t\in J,\\ D_qv(0)=m,\ D_q^2v(0)=D_q^3v(0)=\cdots=D_q^{n-1}v(0)=0,\\ v(\eta)=\int_0^{+\infty}w(t)v(t)d_qt+d, \end{cases}$$

类似引理 3 的求解过程，上述方程有唯一解

$$v(t)=\int_0^{+\infty}U(t,qs)y(s)d_qs+\psi(t)+\frac{d}{\Lambda}\geq 0,$$

$$D_qv(t)=\frac{1}{\Gamma_q(\alpha-1)}\int_0^t(t-qs)^{(\alpha-2)}y(s)d_qs+\lambda\geq 0\ 。$$

定理 1 假设 (C_1)-(C_3) 成立，若方程（1）有上解 $v_0=v_0(t)$，则（1）至少有一个正解 $v^*=v^*(t)$ 且

$$0\leq v^*(t)\leq v_0(t),\ D_qv^*(t)\geq 0,\ t\in J\ 。$$

证明 若 $v_1=v_1(t)$ 是方程

$$\begin{cases} {}^cD_q^{\alpha}v_1(t)=g(t,v_0(t),D_qv_0(t),v_0(v_0(t))),\ t\in J,\\ D_qv_1(0)=\lambda,\ D_q^2v_1(0)=D_q^3v_1(0)=\cdots=D_q^{n-1}v_1(0)=0,\\ v_1(\eta)=\int_0^{+\infty}w(t)v_1(t)d_qt \end{cases}\tag{7}$$

的唯一解，则由引理 3 可知 $v_1(t)=\int_0^{+\infty}U(t,qs)g(s,v_0(s),D_qv_0(s),v_0(v_0(s)))d_qs+\psi(t)$ 。

由定义 1 和(7)可得

$$\begin{cases} {}^cD_q^{\alpha}\left(v_0(t)-v_1(t)\right)\geq 0,\ t\in J,\\ D_q\left(v_0(t)-v_1(0)\right)\geq 0,\ D_q^2v_0(0)-D_q^2v_1(0)=D_q^3v_0(0)-D_q^3v_1(0)=\cdots=D_q^{n-1}v_0(0)-D_q^{n-1}v_1(0)=0,\\ v_0(\eta)-v_1(\eta)\geq\int_0^{+\infty}w(t)\left(v_0(t)-v_1(t)\right)d_qt\ , \end{cases}$$

根据引理 8 可得 $v_0(t)-v_1(t)\geq 0,\ D_q\left(v_0(t)-v_1(0)\right)\geq 0$ ，即 $v_1(t)\leq v_0(t)\in P$ 。

由假设 (C_1) (C_2) 和（7）有

$$\begin{cases} {}^cD_q^{\alpha}v_1(t)=g(t,v_0(t),D_qv_0(t),v_0(v_0(t)))\geq g(t,v_1(t),D_qv_1(t),v_1(v_1(t)))\ ,\ t\in J\ ,\\ D_qv_1(0)\geq\lambda\ ,\ D_q^2v_1(0)=D_q^3v_1(0)=\cdots=D_q^{n-1}v_1(0)=0\ ,\\ v_1(\eta)\geq\int_0^{+\infty}w(t)v_1(t)d_qt\ , \end{cases}$$

且 $0\leq\int_0^{+\infty}Q(s)\,{}^cD_q^{\alpha}v_1(s)d_qs=\int_0^{+\infty}Q(s)g(t,v_0(s),D_qv_0(s),v_0(v_0(s)))d_qs\leq\int_0^{+\infty}Q(s)\,{}^cD_q^{\alpha}v_0(s)d_qs<+\infty$ ，根据上解的定义可知， $v_1=v_1(t)$ 是方程（1）的上解。

将 $v_0=v_0(t)$ 作为起始点，建立（1）的一个上解序列 $\{v_i\}\subset P$ ，

$$v_{i+1}=v_{i+1}(t)=(Tv_i)(t)=U(t,qs)g(s,v_i(s),D_qv_i(s),v_i(v_i(s)))d_qs+\psi(t)\ ,\ \ i=1,2\cdots\ 。$$

易证 $0\leq\cdots\leq v_{i+1}\leq v_i\leq\cdots\leq v_2\leq v_1\leq v_0$，即 $\{v_i\}\subset P$ 是单调递减的有界序列。由算子 T 的全连续性可知，存在 $v^*\in P$ 使得 $\lim\limits_{i\to\infty}v_i=v^*$ ，且 $v^*=Tv^*$ 。换句话说， v^* 是算子 T 的一个不动点，且 $v^*\geq 0$ ， $D_qv^*(t)\geq 0$ ，则方程

（1）有一个正解 $v^*=v^*(t)$ 且是单调递减的，更进一步说若 $\lambda\neq 0$ 或 $\forall t\in J, g(t,x,y,z)\not\equiv 0$ 则 $v^*=v^*(t)$ 是（1）的一个非平凡解。

3 应用举例

考虑迭代泛函分数阶 $q-$差分方程边值问题

$$\begin{cases} {}^{c}D_{0.5}^{2.5}v(t)-g(t,v(t),D_{0.5}v(t),v(v(t)))=0,\ t\in[0,+\infty), \\ D_q v(0)=1,\ D_q^2 v(0)=D_q^3 v(0)=\cdots=D_q^{n-1}v(0)=0,\ v(\frac{1}{15})=\int_0^{+\infty}w(t)v(t)d_q t\ , \end{cases} \tag{8}$$

其中 $q=0.5,\alpha=2.5,g(t,x,y,z)=e^{-t}\ln(1+x)+2e^{-t}y+e^{-2t}z,\lambda=\frac{3}{4},w(t)=e^{-3t},\eta=\frac{1}{15}$。

对于 $t\in[0,+\infty),\ x_i,y_i,z_i\in[0,+\infty),i=1,2,$

$$0\le g(t,x_2,y_2,z_2)-g(t,x_1,y_1,z_1)，\text{其中 } x_1\le x_2,\ y_1\le y_2,\ z_1\le z_2,$$

取 $l_1(t)=e^{-\frac{t}{2}}$，$l_2(t)=3e^{-t}$，$l_3(t)=e^{-t}$，可以得到

$$\int_0^{+\infty}l_1(s)(1+s^{\alpha-1})d_q s=l_1\approx 6.8672<+\infty,\ \int_0^{+\infty}l_2(s)(1+s^{\alpha-2})d_q s=l_2\approx 4.0818<+\infty,$$

$$\int_0^{+\infty}l_3(s)d_q s=l_3\approx 0.7213<+\infty,\ \int_0^{+\infty}l_3(s)(1+s^{\alpha-1})^{\alpha-1}d_q s=l_4\approx 3.1521<+\infty,$$

且 $0\le g(t,x,y,z)\le l_1(t)x+l_2(t)y+l_3(t)z$。因此，条件 (C_1) 和 (C_2) 满足。

通过计算，有

$$0\le\int_0^{+\infty}w(\tau)d_q\tau=\int_0^{+\infty}e^{-3\tau}d_q\tau\approx 0.2404<1,\ 0\le\int_0^{+\infty}\tau^{\alpha-1}w(\tau)d_q\tau=\int_0^{+\infty}\tau^{\frac{3}{2}}e^{-3\tau}d_q\tau\approx 0.0615<+\infty,$$

$$\eta^{\alpha-1}=(\frac{1}{15})^{1.5}\approx 0.0172\le\int_\eta^{+\infty}(\tau-\eta)^{(\alpha-1)}w(\tau)d_q\tau=\int_{\frac{1}{15}}^{+\infty}(\tau-\frac{1}{15})^{(1.5)}e^{-3\tau}d_q\tau\approx 0.0504<+\infty,$$

$$\eta=\frac{1}{15}\le\int_0^{+\infty}\tau w(\tau)d_q\tau=\int_0^{+\infty}\tau e^{-3\tau}d_q\tau\approx 0.0801<+\infty,$$

即条件 (C_3) 成立。

取初值 $v_0(t)=3t^{\alpha}+t+\frac{1}{2}=3t^{2.5}+t+\frac{1}{2}$，则 ${}^{c}D_q^{\alpha}v_0(t)=3\Gamma_q(3.5)\approx 5.8806$，

且 $Q(s)=\int_{qs}^{+\infty}(\tau-qs)^{(\alpha-1)}w(\tau)d_q\tau=\int_{qs}^{+\infty}(\tau-qs)^{(1.5)}e^{-3\tau}d_q\tau\approx 0.1695e^{-3s}$，$\int_0^{+\infty}Q(s)\,{}^{c}D_q^{\alpha}v_0(s)d_q s=0.2396<+\infty$，

$h(t):=g(t,v_0(t),D_q v_0(t),v_0(v_0(t)))$

$=e^{-t}\ln(\frac{3}{2}+3t^{2.5}+t)+e^{-t}[3(1-0.5^{2.5})t^{1.5}+1]+e^{-2t}[3(3t^{2.5}+t+\frac{1}{2})^{2.5}+3t^{2.5}+t+1]$ 。

易知，当 $t\in[0,+\infty)$ 时，函数 $h(t)$ 有最大值 $h(0)\approx 2.9357$。

$$\begin{cases} {}^{c}D_q^{\alpha}v_0(t)=3\Gamma_q(3.5)>\max_{t\in J}h(t)=g(t,v_0(t),D_q v_0(t),v_0(v_0(t))),\ t\in J, \\ D_q v_0(0)=1>\frac{3}{4}=\lambda,\ D_q^2 v_0(0)=D_q^3 v_0(0)=\cdots=D_q^{n-1}v_0(0)=0, \\ v_0(\eta)=v_0(\frac{1}{15})=3\cdot(\frac{1}{15})^{2.5}+\frac{1}{15}+\frac{1}{2}\approx 0.5701\ge\int_0^{+\infty}w(t)v_0(t)d_q t\approx 0.3542, \end{cases}$$

则 $v_0=v_0(t)$ 是上述边值问题的上解。

因此，根据定理 1 可知，边值问题（8）至少有一个解 $v^*=v^*(t)$。

4 结语

迭代泛函量子差分方程的研究对于丰富 $q-$差分方程的理论至关重要，本文运用上下解法和单调迭代技术，探讨了一类无穷区间上迭代泛函高阶分数阶 $q-$差分方程边值问题的可解性，得到了无穷区间上迭代泛函分数阶 $q-$差分方程的最大正解。值得注意的是，当 $q\to 1^-$ 时，方程可退化为文献[22]研究的方程并包含其结果。因此，本文的研究结果不仅丰富了非线性分数阶量子差分方程可解性理论，而且为这类方程在生物工

程等科学领域的应用提供了理论指导。未来，将利用 Krasnosel'skii 不动点定理以及一些新的不动点定理研究迭代泛函分数阶 $q-$差分方程非局部问题、无穷区间共振问题等的可解性，并尝试研究其数值解法。

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