文章编号：1008-1542（2024）00-0000-00

无穷区间上带 $p-$Laplacian 算子的分数阶 $q-$差分方程的迭代正解

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摘　要：为了丰富分数阶 $q-$差分方程边值问题的基本理论，研究了一类无穷区间上带 $p-$Laplacian 算子的非线性分数阶 $q-$差分方程边值问题。首先，计算线性分数阶 $q-$差分方程边值问题的 Green 函数并研究其性质；其次，引入无穷区间上的紧性判定准则并在抽象空间上构造积分算子；再次，选取初值函数，运用单调迭代技巧，获得边值问题正解的存在性；最后，通过实例验证所得结果的有效性。结果表明，在赋予非线性项 f 一定的增长条件下，通过构造迭代序列，可得到分数阶 $q-$差分方程的最大和最小正解。研究结果拓展了已有的相关结论，并为分数阶 $q-$差分方程在数学、物理等领域的进一步应用提供了重要的理论依据。

关键词：非线性泛函分析；分数阶 $q-$差分方程；$p-$Laplacian 算子；单调迭代技巧；无穷区间

中图分类号：O175.8　　　　　　　　文献标识码：A

Iterative positive solutions for fractional *q*-difference equations with *p*-Laplacian operator on infinite intervals

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Abstract: In order to enrich the basic theory of boundary value problems for fractional *q*-difference equations, we investigate the boundary value problems for a class of nonlinear fractional *q*-difference equations with *p*-Laplacian operator on infinite intervals. Firstly, the Green function of the boundary value problem of linear fractional *q*-difference equation is calculated and its properties are studied. Secondly, we introduce the compactness criterion on infinite intervals and construct the integral operator on an abstract space. Thirdly, by selecting the initial value functions and using the monotone iterative technique, we acquire the existence of positive solutions for the boundary value problem. Finally, an example is provided to illustrate the validity of obtained results. The results show that when certain increasing condition is given to the nonlinear term *f*, we can obtain the maximum and minimum positive solutions of the fractional *q*-difference equation by establishing iterative sequences. The research results extend the existing relevant conclusions and provide an important theoretical basis for the further application of fractional *q*-difference equations in mathematics, physics and other fields.

Keywords: nonlinear functional analysis; fractional *q*-difference equations; *p*-Laplacian operator; monotone iterative technique; infinite intervals

收稿日期：2024-03-04；修回日期：2024-04-30；责任编辑：张士莹
基金项目：国家重点研发计划项目（2022YFB3806000）；国家自然科学基金（12272011）；河北省自然科学基金（A2015208114）；河北省教育厅基金（QN2017063）
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量子微积分，又称$q-$微积分，基本定义最早由 JACKSON[1]建立，广泛应用于超几何级数、复分析、粒子物理、量子力学等领域。20 世纪中期，AL-SALAM[2]和 AGARWAL[3]进一步提出了分数阶$q-$微积分的概念。与经典的$q-$微积分相比，分数阶$q-$微积分能更准确地描述各种材料和过程中的记忆和遗传等现象[4-8]。随着分数阶$q-$微积分理论的不断发展，人们对分数阶$q-$差分方程的研究日益深入。近年来，关于分数阶$q-$差分方程边值问题的可解性取得了许多优秀的成果[9-17]。

与此同时，带$p-$Laplacian 算子的分数阶微分方程作为非牛顿力学、弹性理论、天体物理学和经典电学等诸多领域中的重要数学模型备受关注。众多学者研究了有限区间上包含各类边界条件的带$p-$Laplacian 算子的分数阶微分方程[18-21]。事实上，与有限区间相比，无穷区间上的边值问题在现实生活中更为常见。例如：它们自然地出现在非线性椭圆方程径向对称解、半无穷多孔介质气压模型、不稳定气流流速等系列问题中[22-25]。当前，关于无穷区间上带$p-$Laplacian 算子的分数阶边值问题研究正处于快速发展阶段。2015 年，WANG 等[26]利用单调迭代技巧研究了一类高阶分数阶微分方程

$$\begin{cases} D_{0^+}^{\beta}\left(\phi_p\left(D_{0^+}^{\alpha}x(t)\right)\right)+a(t)f\left(t,x(t),(Tx)(t),(Sx)(t)\right)=0,\ t\in J, \\ x(0)=x'(0)=\cdots=x^{(n-2)}(0)=0, \\ D_{0^+}^{\alpha}x(0)=0,\quad \lim\limits_{t\to+\infty}D_{0^+}^{\alpha-1}x(t)=\int_0^{+\infty}h(t)x(t)dt \end{cases}$$

正解的存在性。其中，$J:=[0,+\infty)$；$0<\beta\le 1$；$n-1<\alpha\le n$；$n\in\mathbb{N}$且$n\ge 2$；$\phi_p(s)=|s|^{p-2}s(p>1)$；$D_{0^+}^{\alpha}$和$D_{0^+}^{\beta}$是 Riemann-Liouville 型分数阶导数；$a(t)\in L(J,J)$；$f\in C(J\times J\times J\times J,J)$；$h(t)\in L(J,J)$。

2021 年，FENIZRI 等[27]利用 Krasnoselskii 不动点定理，研究了一类具有积分边界条件的分数阶边值问题

$$\begin{cases} D_{0^+}^{\delta}\left(\phi_p\left(D_{0^+}^{\alpha}x(t)\right)\right)+a(t)f\left(t,x(t),D_{0^+}^{\alpha-1}(t)\right)=0,\ t\in J, \\ x(0)=0,\ I_{0^+}^{1-\delta}\left(\phi_p\left(D_{0^+}^{\alpha}x(0)\right)\right)=0, \\ D_{0^+}^{\alpha-2}x(0)=\int_0^{+\infty}h(t)x(t)dt,\ D_{0^+}^{\alpha-1}x(+\infty)=\int_0^{+\infty}\mathrm{g}(t)x(t)dt \end{cases}$$

解的存在性。其中，$0<\delta\le 1$；$2<\alpha\le 3$；$a(t)\in L(J,J)$；$f\in C(J\times\mathbb{R}^2,\mathbb{R})$；$g(t)$和$h(t)$均为非负函数。

值得注意的是，目前对于无穷区间上带$p-$Laplacian 算子的分数阶$q-$差分方程的研究尚显不足，特别是当该类问题的非线性项包含未知函数的任意低阶分数阶$q-$导数时，相关结果更为有限。基于上述情况，本文研究如下边值问题

$$\begin{cases} D_q^{\beta}\left(\phi_p\left(D_q^{\alpha}u(t)\right)\right)+a(t)f\left(t,u(t),(Q_1u)(t),(Q_2u)(t),D_q^{\gamma_1}u(t),\cdots,D_q^{\gamma_{n-1}}u(t)\right)=0,\ t\in J, \\ u(0)=D_qu(0)=\cdots=D_q^{n-2}u(0)=0, \\ \phi_p\left(D_q^{\alpha}u(0)\right)=D_q\left(\phi_p\left(D_q^{\alpha}u(0)\right)\right)=\cdots=D_q^{n-2}\left(\phi_p\left(D_q^{\alpha}u(0)\right)\right)=0, \\ D_q^{\alpha-1}u(+\infty)=\int_0^{+\infty}g(t)u(t)d_qt \end{cases}\tag{1}$$

正解的存在性。其中，$0<q<1$；$0<\beta\le n-1$；$n-1<\alpha\le n$；$n-i<\alpha-\gamma_i\le n-i+1$；$n\in\mathbb{N}$且$n\ge 2$；$D_q^{(\cdot)}$是 Riemann-Liouville 型分数阶$q-$导数；$\phi_p(s)=|s|^{p-2}s(p>1)$；$a(t)\in C(J,J)$；$f\in C(J\times\mathbb{R}^{n+2},J)$；$g(t)$是非负函数且满足$\int_0^{+\infty}g(t)t^{\alpha-1}d_qt:=\Lambda<\Gamma_q(\alpha)$。

此外，$(Q_1u)(t)=\int_0^tH_1(t,s)u(s)d_qs$，$(Q_2u)(t)=\int_0^{+\infty}H_2(t,s)u(s)d_qs$。其中，$H_1(t,s)\in C(D,J)$；$D=\{(t,s)\in J\times J,t\ge s\}$；$H_2(t,s)\in C(J\times J,J)$。

本文用到的假设如下：

(A_1) $f\in C(J\times\mathbb{R}^{n+2},J)$，$f(t,0,\cdots,0)\ne 0$，且当变量$u_1,u_2,u_3,v_1,\cdots,v_{n-1}$有界时，对于任意的$t\in J$，函数$f\left(t,\left(1+t^{\alpha-1}\right)u_1,u_2,u_3,\left(1+t^{\alpha-\gamma_1-1}\right)v_1,\cdots,\left(1+t^{\alpha-\gamma_{n-1}-1}\right)v_{n-1}\right)$是有界的；

(A_2) $a(t)\in C(J,J)$，$a(t)\ne 0$，且

$$0<\int_0^{+\infty}\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)d_q\tau\right)d_qs=K_p<+\infty;$$

(A_3) $H_1(t,s)$，$H_2(t,s)\in L_q[0,+\infty)$，且

$$0<\sup_{t\in J}\int_0^t H_1(t,s)(1+s^{\alpha-1})d_q s=h_1^*<+\infty,$$

$$0<\sup_{t\in J}\int_0^{+\infty} H_2(t,s)(1+s^{\alpha-1})d_q s=h_2^*<+\infty;$$

(A_4) 对于任意的 $t\in J$，函数 $f\left(t,u_1,u_2,u_3,v_1,\cdots,v_{n-1}\right)$ 关于变量 $u_1,u_2,u_3,v_1,\cdots,v_{n-1}$ 是递增的。

1 预备知识

定义 1[3] 设 $\alpha\ge 0$，函数 f 定义在 $[0,1]$ 上，则 Riemann-Liouville 型分数阶 $q-$积分为 $I_q^0 f(t)=f(t)$，且

$$(I_q^\alpha f)(t)=\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}f(s)d_q s,\quad \alpha>0,\ t\in[0,1]\text{。}$$

定义 2[4] 设 $\alpha\ge 0$，函数 f 定义在 $[0,1]$ 上，则 Riemann-Liouville 型分数阶 $q-$导数为 $D_q^0 f(t)=f(t)$，且

$$(D_q^\alpha f)(t)=(D_q^n I_q^{n-\alpha} f)(t),\quad \alpha>0,$$

其中，n 是大于等于 α 的最小整数。

引理 1[4] 设 $\alpha,\beta\ge 0$，函数 f 定义在 $[0,1]$ 上，则下列等式成立：

(i) $(I_q^\beta I_q^\alpha f)(t)=(I_q^{\alpha+\beta} f)(t)$；

(ii) $(D_q^\alpha I_q^\alpha f)(t)=f(t)$；

(iii) $I_q^\beta t^\lambda=\dfrac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\beta+1)}t^{\lambda+\beta}$，$\lambda>-1$。

引理 2[5] 设 $\alpha\ge 0$，n 是一个正整数，则下式成立：

$$(I_q^\alpha D_q^n f)(t)=(D_q^n I_q^\alpha f)(t)-\sum_{k=0}^{n-1}\frac{t^{\alpha-n+k}}{\Gamma_q(\alpha-n+k+1)}(D_q^k f)(0)\text{。}$$

引理 3 若函数 $h(t)\in L_q[0,+\infty)$，则线性分数阶 $q-$差分方程边值问题

$$\begin{cases} D_q^\beta\left(\phi_p\left(D_q^\alpha u(t)\right)\right)+h(t)=0,\quad t\in J,\\ u(0)=D_q u(0)=\cdots=D_q^{n-2}u(0)=0,\\ \phi_p\left(D_q^\alpha u(0)\right)=D_q\left(\phi_p\left(D_q^\alpha u(0)\right)\right)=\cdots=D_q^{n-2}\left(\phi_p\left(D_q^\alpha u(0)\right)\right)=0,\\ D_q^{\alpha-1}u(+\infty)=\int_0^{+\infty} g(t)u(t)d_q t \end{cases}\tag{2}$$

有唯一解

$$u(t)=\zeta_p\int_0^{+\infty} G(t,qs)\phi_p^{-1}\left(\int_0^s (s-q\tau)^{(\beta-1)}h(\tau)d_q\tau\right)d_q s,\tag{3}$$

其中

$$\zeta_p=\left(\Gamma_q(\beta)\right)^{\frac{1}{1-p}},\quad G(t,qs)=G_1(t,qs)+G_2(t,qs),$$

且

$$G_1(t,qs)=\frac{1}{\Gamma_q(\alpha)}\begin{cases} t^{\alpha-1}-(t-qs)^{(\alpha-1)}, & 0\le qs\le t<+\infty,\\ t^{\alpha-1}, & 0\le t\le qs<+\infty,\end{cases}$$

$$G_2(t,qs)=\frac{t^{\alpha-1}}{\Gamma_q(\alpha)-\Lambda}\int_0^{+\infty} g(t)G_1(t,qs)d_q t\text{。}$$

证明：对分数阶 $q-$差分方程(2)两边同时积分，由引理 2 可得

$$\phi_p\left(D_q^\alpha u(t)\right)=-I_q^\beta h(t)+c_1 t^{\beta-1}+c_2 t^{\beta-2}+\cdots+c_{n-1}t^{\beta-(n-1)},$$

其中 $c_1,c_2,\cdots,c_{n-1}$ 是待定系数。由边界条件 $\phi_p\left(D_q^\alpha u(0)\right)=D_q\left(\phi_p\left(D_q^\alpha u(0)\right)\right)=\cdots=D_q^{n-2}\left(\phi_p\left(D_q^\alpha u(0)\right)\right)=0$ 可得，$c_1=c_2=\cdots=c_{n-1}=0$。因此，

$$\phi_p\left(D_q^\alpha u(t)\right)=-\frac{1}{\Gamma_q(\beta)}\int_0^t (t-q\tau)^{(\beta-1)}h(\tau)d_q\tau,$$

且

$$D_q^{\alpha}u(t)=-\phi_p^{-1}\left(\frac{1}{\Gamma_q(\beta)}\int_0^t(t-q\tau)^{(\beta-1)}h(\tau)d_q\tau\right)。$$

又由引理 2，有

$$u(t)=-\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}\phi_p^{-1}\left(\frac{1}{\Gamma_q(\beta)}\int_0^s(s-q\tau)^{(\beta-1)}h(\tau)d_q\tau\right)d_qs$$
$$+d_1t^{\alpha-1}+d_2t^{\alpha-2}+\cdots+d_nt^{\alpha-n}。$$

令 $\psi(s)=\phi_p^{-1}\left(\frac{1}{\Gamma_q(\beta)}\int_0^s(s-q\tau)^{(\beta-1)}h(\tau)d_q\tau\right)$。由边界条件 $u(0)=D_q^j u(0)=0$，$j=1,\cdots,n-1$，和 $D_q^{\alpha-1}u(+\infty)=\int_0^{+\infty}g(t)u(t)d_qt$，可得

$$d_1=\frac{1}{\Gamma_q(\alpha)}\left(\int_0^{+\infty}g(t)u(t)d_qt+\int_0^{+\infty}\psi(s)d_qs\right),\ d_2=d_3=\cdots=d_n=0。$$

此时，

$$\begin{aligned}u(t)&=-\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\psi(s)d_qs+\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\left(\int_0^{+\infty}g(t)u(t)d_qt+\int_0^{+\infty}\psi(s)d_qs\right)\\&=\int_0^{+\infty}G_1(t,qs)\psi(s)d_qs+\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^{+\infty}g(t)u(t)d_qt\ 。\end{aligned}\tag{4}$$

在式(4)等号两边同时乘以 g(t)，并从 0 到 $+\infty$ 积分，可得

$$\int_0^{+\infty}g(t)u(t)d_qt=\int_0^{+\infty}g(t)\left(\int_0^{+\infty}G_1(t,qs)\psi(s)d_qs\right)d_qt+\frac{\Lambda}{\Gamma_q(\alpha)}\int_0^{+\infty}g(t)u(t)d_qt，$$

因此，

$$\int_0^{+\infty}g(t)u(t)d_qt=\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha)-\Lambda}\int_0^{+\infty}g(t)\left(\int_0^{+\infty}G_1(t,qs)\psi(s)d_qs\right)d_qt\ 。\tag{5}$$

将式(5)代入式(4)，有

$$\begin{aligned}u(t)&=\int_0^{+\infty}G_1(t,qs)\psi(s)d_qs+\frac{t^{\alpha-1}}{\Gamma_q(\alpha)-\Lambda}\int_0^{+\infty}g(t)\left(\int_0^{+\infty}G_1(t,qs)\psi(s)d_qs\right)d_qt\\&=\int_0^{+\infty}G_1(t,qs)\psi(s)d_qs+\frac{t^{\alpha-1}}{\Gamma_q(\alpha)-\Lambda}\int_0^{+\infty}\psi(s)\left(\int_0^{+\infty}g(t)G_1(t,qs)d_qt\right)d_qs\\&=\int_0^{+\infty}G_1(t,qs)\psi(s)d_qs+\int_0^{+\infty}G_2(t,qs)\psi(s)d_qs\\&=\zeta_p\int_0^{+\infty}G(t,qs)\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}h(\tau)d_q\tau\right)d_qs,\end{aligned}$$

证毕。

引理 4 $\forall(t,s)\in J\times J$，函数 $G(t,qs)$ 是连续的，且具有以下性质：

$$0\le G(t,qs)\le\frac{t^{\alpha-1}}{\Gamma_q(\alpha)-\Lambda},\quad 0\le\frac{G(t,qs)}{1+t^{\alpha-1}}\le\frac{1}{\Gamma_q(\alpha)-\Lambda}。$$

证明： $\forall(t,s)\in J\times J$，函数 $G(t,qs)$ 显然连续，由引理 3 得

$$0\le G_1(t,qs)\le\frac{t^{\alpha-1}}{\Gamma_q(\alpha)},$$

$$0\le G_2(t,qs)\le\frac{t^{\alpha-1}}{\Gamma_q(\alpha)-\Lambda}\int_0^{+\infty}g(t)\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}d_qt=\frac{\Lambda t^{\alpha-1}}{\Gamma_q(\alpha)\left(\Gamma_q(\alpha)-\Lambda\right)}。$$

因此，

$$0\le G(t,qs)\le\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}+\frac{\Lambda t^{\alpha-1}}{\Gamma_q(\alpha)\left(\Gamma_q(\alpha)-\Lambda\right)}=\frac{t^{\alpha-1}}{\Gamma_q(\alpha)-\Lambda},$$

且

$$0 \le \frac{G(t,qs)}{1+t^{\alpha-1}} \le \frac{t^{\alpha-1}}{\left(\Gamma_q(\alpha)-\Lambda\right)(1+t^{\alpha-1})} \le \frac{1}{\Gamma_q(\alpha)-\Lambda},$$

证毕。

注 1 设$u(t)$是边值问题(2)的一个解，经简单计算，有

$$D_q^{\gamma_i}u(t) = \zeta_p \int_0^{+\infty} G_i^*(t,qs)\phi_p^{-1}\left(\int_0^s (s-q\tau)^{(\beta-1)}h(\tau)d_q\tau\right)d_q s,\ \ i=1,\cdots,n-1,$$

其中

$$G_i^*(t,qs) = G_{i1}^*(t,qs) + G_{i2}^*(t,qs),$$

且

$$G_{i1}^*(t,qs) = \frac{1}{\Gamma_q(\alpha-\gamma_i)}\begin{cases} t^{\alpha-\gamma_i-1} - (t-qs)^{(\alpha-\gamma_i-1)}, & 0 \le qs \le t < +\infty, \\ t^{\alpha-\gamma_i-1}, & 0 \le t \le qs < +\infty, \end{cases}$$

$$G_{i2}^*(t,qs) = \frac{\Gamma_q(\alpha)t^{\alpha-\gamma_i-1}}{\Gamma_q(\alpha-\gamma_i)\left(\Gamma_q(\alpha)-\Lambda\right)}\int_0^{+\infty} g(t)G_1(t,qs)d_q t 。$$

注 2 $\forall(t,s)\in J\times J$，函数$G_i^*(t,qs)$是连续的，且具有以下性质：

$$0 \le G_i^*(t,qs) \le \frac{\Gamma_q(\alpha)t^{\alpha-\gamma_i-1}}{\Gamma_q(\alpha-\gamma_i)\left(\Gamma_q(\alpha)-\Lambda\right)},\ \ i=1,\cdots,n-1,$$

$$0 \le \frac{G_i^*(t,qs)}{1+t^{\alpha-\gamma_i-1}} \le \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma_i)\left(\Gamma_q(\alpha)-\Lambda\right)},\ \ i=1,\cdots,n-1 。$$

考虑空间

$$E = \left\{u \middle| u\in C(J),\ D_q^{\gamma_i}u \in C(J), \left\|\frac{u}{1+t^{\alpha-1}}\right\|_\infty < +\infty, \left\|\frac{D_q^{\gamma_i}u}{1+t^{\alpha-\gamma_i-1}}\right\|_\infty < +\infty,\ i=1,\cdots,n-1\right\},,$$

并定义其范数为

$$\|u\|_E = \max\left\{\left\|\frac{u}{1+t^{\alpha-1}}\right\|_\infty, \left\|\frac{D_q^{\gamma_i}u}{1+t^{\alpha-\gamma_i-1}}\right\|_\infty,\ i=1,\cdots,n-1\right\},$$

其中，$\|u(t)\|_\infty = \sup_{t\in J}|u(t)|$。类似于文献[21]中引理 8 的证明方法，不难证明E是一个 Banach 空间。此外，由于 Arzela-Ascoli 定理在无穷区间上是失效的，所以引入如下紧性判定准则。

引理 5 设$B\subset E$是一个有界集，则B是E中的相对紧集当且仅当其满足以下条件：

(i) $\left\{\frac{u(t)}{1+t^{\alpha-1}} \middle| u\in B\right\}, \left\{\frac{D_q^{\gamma_i}u(t)}{1+t^{\alpha-\gamma_i-1}} \middle| u\in B\right\}$，$i=1,\cdots,n-1$在$J$的任意紧子区间上是等度连续的；

(ii) $\forall\varepsilon>0, \exists C=C(\varepsilon)>0$，使得$\forall t_1,t_2\ge C,\ u\in B$，有$\left|\frac{u(t_2)}{1+t_2^{\alpha-1}} - \frac{u(t_1)}{1+t_1^{\alpha-1}}\right| < \varepsilon$，$\left|\frac{D_q^{\gamma_i}u(t_2)}{1+t_2^{\alpha-\gamma_i-1}} - \frac{D_q^{\gamma_i}u(t_1)}{1+t_1^{\alpha-\gamma_i-1}}\right| < \varepsilon$，$i=1,\cdots,n-1$成立。

证明：根据条件，只需证明B是完全有界的。

1)考虑区间$t\in[0,I]$，定义 Banach 空间$B_{[0,I]} = \left\{u(t) \middle| u(t)\in B, t\in[0,I]\right\}$，其范数为$\|u(t)\| = \sup_{t\in[0,I]}\left|\frac{u(t)}{1+t^{\alpha-1}}\right|$。由 Arzela-Ascoli 定理和条件 i)可得，$B_{[0,I]}$在E中是相对紧的，因此$B_{[0,I]}$是完全有界的。那么，$\forall\varepsilon>0$，存在有限个开球$B_\varepsilon(u_j)$，使得$B_{[0,I]} \subset \bigcup_{j=1}^m B_\varepsilon(u_j)$，其中

$$B_\varepsilon(u_j) = \left\{u(t)\in B_{[0,I]} : \|u-u_j\| = \sup_{t\in[0,I]}\left|\frac{u(t)}{1+t^{\alpha-1}} - \frac{u_j(t)}{1+t^{\alpha-1}}\right| < \varepsilon\right\} 。$$

类似地，定义 Banach 空间$B_{[0,I]}^{\gamma_i} = \left\{D_q^{\gamma_i}u(t) \middle| u(t)\in B_{[0,I]}\right\}$，其范数为$\left\|D_q^{\gamma_i}u(t)\right\| = \sup_{t\in[0,I]}\left|\frac{D_q^{\gamma_i}u(t)}{1+t^{\alpha-\gamma_i-1}}\right|$。同理可得，存在有限个开球$B_\varepsilon(D_q^{\gamma_i}u_k)$，使得$B_{[0,I]}^{\gamma_i} \subset \bigcup_{k=1}^l B_\varepsilon(D_q^{\gamma_i}u_k)$，其中

$$B_\varepsilon(D_q^{\gamma_i}u_k)=\left\{D_q^{\gamma_i}u_k\in B_{[0,I]}^{\gamma_i}：\ \left\|D_q^{\gamma_i}u-D_q^{\gamma_i}u_k\right\|=\sup_{t\in[0,I]}\left|\frac{D_q^{\gamma_i}u}{1+t^{\alpha-\gamma_i-1}}-\frac{D_q^{\gamma_i}u_k}{1+t^{\alpha-\gamma_i-1}}\right|<\varepsilon\right\}。$$

2) 定义 $B_{jk}=\left\{u(t)\in B\middle|u_{[0,I]}\in B_\varepsilon(u_j),D_q^{\gamma_i}u_{[0,I]}\in B_\varepsilon(D_q^{\gamma_i}u_k)\right\}$，显然 $B_{[0,I]}\subset\bigcup_{\substack{1\le j\le m,\\1\le k\le l}}B_{jk[0,I]}$。取 $u_{jk}\in B_{jk}$，由上述证明中的 1)可得，对于 $u(t)\in B$，存在 j 和 k，使得 $u_{[0,I]}\in B_\varepsilon(u_j)$，$D_q^{\gamma_i}u_{[0,I]}\in B_\varepsilon(D_q^{\gamma_i}u_k)$。因此，当 $t\in[0,I]$ 时，有

$$\left|\frac{u(t)}{1+t^{\alpha-1}}-\frac{u_{jk}(t)}{1+t^{\alpha-1}}\right|\le\left|\frac{u(t)}{1+t^{\alpha-1}}-\frac{u_j(t)}{1+t^{\alpha-1}}\right|+\left|\frac{u_j(t)}{1+t^{\alpha-1}}-\frac{u_{jk}(t)}{1+t^{\alpha-1}}\right|<2\varepsilon\,, \tag{6}$$

$$\left|\frac{D_q^{\gamma_i}u(t)}{1+t^{\alpha-\gamma_i-1}}-\frac{D_q^{\gamma_i}u_{jk}(t)}{1+t^{\alpha-\gamma_i-1}}\right|\le\left|\frac{D_q^{\gamma_i}u(t)}{1+t^{\alpha-\gamma_i-1}}-\frac{D_q^{\gamma_i}u_j(t)}{1+t^{\alpha-\gamma_i-1}}\right|+\left|\frac{D_q^{\gamma_i}u_j(t)}{1+t^{\alpha-\gamma_i-1}}-\frac{D_q^{\gamma_i}u_{jk}(t)}{1+t^{\alpha-\gamma_i-1}}\right|<2\varepsilon\,。 \tag{7}$$

当 $t\in[I,+\infty)$ 时，由条件(ii)和式(6)可得

$$\begin{aligned}\left|\frac{u(t)}{1+t^{\alpha-1}}-\frac{u_{jk}(t)}{1+t^{\alpha-1}}\right|&\le\left|\frac{u(t)}{1+t^{\alpha-1}}-\frac{u(I)}{1+t^{\alpha-1}}\right|+\left|\frac{u(I)}{1+t^{\alpha-1}}-\frac{u_{jk}(I)}{1+t^{\alpha-1}}\right|\\&\quad+\left|\frac{u_{jk}(I)}{1+t^{\alpha-1}}-\frac{u_{jk}(t)}{1+t^{\alpha-1}}\right|\\&<\varepsilon+2\varepsilon+\varepsilon\\&=4\varepsilon,\end{aligned} \tag{8}$$

同时，由条件(ii)和式(7)得

$$\begin{aligned}\left|\frac{D_q^{\gamma_i}u(t)}{1+t^{\alpha-\gamma_i-1}}-\frac{D_q^{\gamma_i}u_{jk}(t)}{1+t^{\alpha-\gamma_i-1}}\right|&\le\left|\frac{D_q^{\gamma_i}u(t)}{1+t^{\alpha-\gamma_i-1}}-\frac{D_q^{\gamma_i}u(I)}{1+t^{\alpha-\gamma_i-1}}\right|+\left|\frac{D_q^{\gamma_i}u(I)}{1+t^{\alpha-\gamma_i-1}}-\frac{D_q^{\gamma_i}u_{jk}(I)}{1+t^{\alpha-\gamma_i-1}}\right|\\&\quad+\left|\frac{D_q^{\gamma_i}u_{jk}(I)}{1+t^{\alpha-\gamma_i-1}}-\frac{D_q^{\gamma_i}u_{jk}(t)}{1+t^{\alpha-\gamma_i-1}}\right|\\&<\varepsilon+2\varepsilon+\varepsilon\\&=4\varepsilon。\end{aligned} \tag{9}$$

结合式(6)—式(9)可知，$\left\|u-u_{jk}\right\|_E<4\varepsilon$，则 B 能被球 $B_{4\varepsilon}(u_{jk})$ 覆盖，其中 $B_{4\varepsilon}(u_{jk})=\left\{u(t)\in B:\ \left\|u-u_{jk}\right\|_E<4\varepsilon\right\}$。于是，$B$ 是完全有界的。证毕。

2　主要结论

定义算子 T：$P\to P$ 为

$$Tu(t)=\zeta_p\int_0^{+\infty}G(t,qs)\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)F_u(\tau)d_q\tau\right)d_qs\,,$$

那么

$$D_q^{\gamma_i}Tu(t)=\zeta_p\int_0^{+\infty}G_i^*(t,qs)\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)F_u(\tau)d_q\tau\right)d_qs,\ i=1,\cdots,n-1\,,$$

其中，$P=\left\{u\in E\middle|u(t)\ge0,\ D_q^{\gamma_i}u(t)\ge0,\ i=1,\cdots,n-1,\ t\in J\right\}$ 是一个锥。为方便起见，记

$$L=\max\left\{\frac{1}{\Gamma_q(\alpha)-\Lambda},\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma_i)\left(\Gamma_q(\alpha)-\Lambda\right)}\right\},\quad i=1,\cdots,n-1$$

$$F_u(\tau)=f\left(\tau,u(\tau),\left(Q_1u\right)(\tau),\left(Q_2u\right)(\tau),D_q^{\gamma_1}u(\tau),\cdots,D_q^{\gamma_{n-1}}u(\tau)\right),$$

显然，$u\in P$ 是边值问题(1)的解当且仅当 u 是算子 T 的不动点。

引理 6　假设 $(A_1)-(A_3)$ 成立，则算子 T：$P\to P$ 是全连续的。

证明：易证 T：$P\to P$ 成立。下面证明 T 连续且是相对紧的。

首先，证明 T 是一致有界的。定义有界集 $U_l=\left\{u(t)\in P:\ \left\|u(t)\right\|_E\le l\right\}$，对于任意的 $u(t)\in U_l$，由引理 4 和注 2，有

$$\left\|\frac{Tu(t)}{1+t^{\alpha-1}}\right\|_\infty=\sup_{t\in J}\left|\zeta_p\int_0^{+\infty}\frac{G(t,qs)}{1+t^{\alpha-1}}\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)F_u(\tau)d_q\tau\right)d_qs\right|$$

$$\le\frac{\zeta_p}{\Gamma_q(\alpha)-\Lambda}\phi_p^{-1}(S_l)\int_0^{+\infty}\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)d_q\tau\right)d_qs \tag{10}$$

$$\le\zeta_pLS_l^{\frac{1}{p-1}}K_p,$$

且

$$\left\|\frac{D_q^{\gamma_i}Tu(t)}{1+t^{\alpha-\gamma_i-1}}\right\|_\infty=\sup_{t\in J}\left|\zeta_p\int_0^{+\infty}\frac{G_i^*(t,qs)}{1+t^{\alpha-\gamma_i-1}}\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)F_u(\tau)d_q\tau\right)d_qs\right|$$

$$\le\frac{\zeta_p\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma_i)\big(\Gamma_q(\alpha)-\Lambda\big)}\phi_p^{-1}(S_l)\int_0^{+\infty}\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)d_q\tau\right)d_qs \tag{11}$$

$$\le\zeta_pLS_l^{\frac{1}{p-1}}K_p,\quad i=1,\cdots,n-1,$$

其中，$K_p=\int_0^{+\infty}\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)d_q\tau\right)d_qs$，$S_l=\sup_{t\in J}\left\{F_u(\tau)\middle|u(\tau)\in U_l,\big((Q_1u)(\tau),(Q_2u)(\tau)\big)\in[0,h_1^*l]\times[0,h_2^*l]\right\}$。

因此，

$$\|Tu(t)\|_E\le\zeta_pLS_l^{\frac{1}{p-1}}K_p,$$

即T是一致有界的。

其次，证明T在J的任意紧子区间J'上是等度连续的。对于任意的$u(t)\in U_l$，当t_1，$t_2\in J'$且$t_1\le t_2$时，有

$$\left|\frac{Tu(t_2)}{1+t_2^{\alpha-1}}-\frac{Tu(t_1)}{1+t_1^{\alpha-1}}\right|\le\zeta_p\int_0^{+\infty}\left|\frac{G(t_2,qs)}{1+t_2^{\alpha-1}}-\frac{G(t_1,qs)}{1+t_1^{\alpha-1}}\right|\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)F_u(\tau)d_q\tau\right)d_qs$$

$$\le\zeta_p\Bigg[\int_0^{+\infty}\left|\frac{G_1(t_2,qs)}{1+t_2^{\alpha-1}}-\frac{G_1(t_1,qs)}{1+t_1^{\alpha-1}}\right|\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)F_u(\tau)d_q\tau\right)d_qs$$

$$+\int_0^{+\infty}\left|\frac{G_2(t_2,qs)}{1+t_2^{\alpha-1}}-\frac{G_2(t_1,qs)}{1+t_1^{\alpha-1}}\right|\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)F_u(\tau)d_q\tau\right)d_qs\Bigg]。$$

记$\varPhi(s)=\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)F_u(\tau)d_q\tau\right)$，由引理 3，当$t_1\to t_2$时，有

$$\int_0^{+\infty}\left|\frac{G_1(t_2,qs)}{1+t_2^{\alpha-1}}-\frac{G_1(t_1,qs)}{1+t_1^{\alpha-1}}\right|\varPhi(s)d_qs$$

$$\le\frac{1}{\Gamma_q(\alpha)}\int_0^{t_1}\left(\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|+\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}-\frac{(t_1-qs)^{(\alpha-1)}}{1+t_1^{\alpha-1}}\right|\right)\varPhi(s)d_qs$$

$$+\frac{1}{\Gamma_q(\alpha)}\int_{t_1}^{t_2}\left(\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|+\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}\right|\right)\varPhi(s)d_qs \tag{12}$$

$$+\frac{1}{\Gamma_q(\alpha)}\int_{t_2}^{+\infty}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\varPhi(s)d_qs\to0,$$

且

$$\int_0^{+\infty}\left|\frac{G_2(t_2,qs)}{1+t_2^{\alpha-1}}-\frac{G_2(t_1,qs)}{1+t_1^{\alpha-1}}\right|\varPhi(s)d_qs\le\frac{1}{\Gamma_q(\alpha)-\Lambda}\int_0^{+\infty}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\varPhi(s)d_qs\to0。 \tag{13}$$

那么，当$t_1\to t_2$时，$\left|\frac{Tu(t_2)}{1+t_2^{\alpha-1}}-\frac{Tu(t_1)}{1+t_1^{\alpha-1}}\right|\to0$。同理可得，$\left|\frac{D_q^{\gamma_i}Tu(t_2)}{1+t_2^{\alpha-\gamma_i-1}}-\frac{D_q^{\gamma_i}Tu(t_1)}{1+t_1^{\alpha--\gamma_i-1}}\right|\to0$，$i=1,\cdots,n-1$。于是，算子$T$是等度连续的。

再证明T是等度收敛的。对于任意的$u(t)\in U_l$，易得

$$\int_0^{+\infty}\phi_p^{-1}\left(\int_0^s (s-q\tau)^{(\beta-1)}a(\tau)F_u(\tau)d_q\tau\right)d_q s \le S_l^{\frac{1}{p-1}}K_p < +\infty\ ,$$

进一步，对于任意的$\varepsilon>0$，存在常数$b>0$，使得

$$\int_b^{+\infty}\phi_p^{-1}\left(\int_0^s (s-q\tau)^{(\beta-1)}a(\tau)F_u(\tau)d_q\tau\right)d_q s < \varepsilon\ 。$$

又因为$\lim\limits_{t\to+\infty}\frac{t^{\sigma-1}}{1+t^{\sigma-1}}=\lim\limits_{t\to+\infty}\frac{(t-qs)^{(\sigma-1)}}{1+t^{\sigma-1}}=1, \sigma\in\{\alpha,\alpha-\gamma_i, i=1,\cdots,n-1\}$，所以存在充分大的$C_1>0$，使得对于任意的$t_1,t_2>C_1$，有

$$\left|\frac{t_2^{\sigma-1}}{1+t_2^{\sigma-1}}-\frac{t_1^{\sigma-1}}{1+t_1^{\sigma-1}}\right|<\varepsilon,\quad \left|\frac{(t_2-qs)^{(\sigma-1)}}{1+t_2^{\sigma-1}}-\frac{(t_1-qs)^{(\sigma-1)}}{1+t_1^{\sigma-1}}\right|<\varepsilon\ 。$$

取$C=\max\{C_1,b\}$，对于任意的$t_1,t_2\ge C$，结合式(12)和式(13)，可得

$$\begin{aligned}\left|\frac{Tu(t_2)}{1+t_2^{\alpha-1}}-\frac{Tu(t_1)}{1+t_1^{\alpha-1}}\right| &\le \zeta_p\left[\frac{1}{\Gamma_q(\alpha)}\int_0^{+\infty}\left(\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|+\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}-\frac{(t_1-qs)^{(\alpha-1)}}{1+t_1^{\alpha-1}}\right|\right)\Phi(s)d_q s\right.\\ &\quad+\frac{1}{\Gamma_q(\alpha)}\int_b^{+\infty}\left(\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|+\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}\right|\right)\Phi(s)d_q s\\ &\quad+\frac{1}{\Gamma_q(\alpha)}\int_b^{+\infty}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\Phi(s)d_q s\\ &\quad\left.+\frac{1}{\Gamma_q(\alpha)-\Lambda}\int_0^{+\infty}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\Phi(s)d_q s\right]\\ &\le \zeta_p\left(\frac{2S_l^{\frac{1}{p-1}}K_p}{\Gamma_q(\alpha)}+\frac{5}{\Gamma_q(\alpha)}+\frac{S_l^{\frac{1}{p-1}}K_p}{\Gamma_q(\alpha)-\Lambda}\right)\varepsilon,\end{aligned}$$

类似地，对于$i=1,\cdots,n-1$，有

$$\left|\frac{D_q^{\gamma_i}Tu(t_2)}{1+t_2^{\alpha-\gamma_i-1}}-\frac{D_q^{\gamma_i}Tu(t_1)}{1+t_1^{\alpha-\gamma_i-1}}\right|\le\zeta_p\left[\frac{2S_l^{\frac{1}{p-1}}K_p}{\Gamma_q(\alpha-\gamma_i)}+\frac{5}{\Gamma_q(\alpha-\gamma_i)}+\frac{\Gamma_q(\alpha)S_l^{\frac{1}{p-1}}K_p}{\Gamma_q(\alpha-\gamma_i)\left(\Gamma_q(\alpha)-\Lambda\right)}\right]\varepsilon\ 。$$

由上可得，算子T是等度收敛的。

最后，证明T：$P\to P$是连续的。设$u_m,u\in P$满足$u_m\to u(m\to\infty)$，根据函数f的连续性及 Lebesgue 控制收敛定理，有

$$\begin{aligned}&\left\|Tu_m(t)-Tu(t)\right\|_E\\ &=\max\left\{\left\|\frac{Tu_m(t)}{1+t^{\alpha-1}}-\frac{Tu(t)}{1+t^{\alpha-1}}\right\|_\infty,\left\|\frac{D_q^{\gamma_i}Tu_m(t)}{1+t^{\alpha-\gamma_i-1}}-\frac{D_q^{\gamma_i}Tu(t)}{1+t^{\alpha-\gamma_i-1}}\right\|_\infty\right\}\\ &\le L\left|\int_0^{+\infty}\phi_p^{-1}\left(\int_0^s (s-q\tau)^{(\beta-1)}a(\tau)F_{u_m}(\tau)d_q s\right)d_q\tau\right.\\ &\quad\left.-\int_0^{+\infty}\phi_p^{-1}\left(\int_0^s (s-q\tau)^{(\beta-1)}a(\tau)F_u(\tau)d_q s\right)d_q\tau\right|\to 0,\ m\to\infty,\end{aligned}$$

因此，算子T是连续的。

综上，由引理 5 可知，算子$T:P\to P$是全连续的。证毕。

定理 1　假设$(A_1)-(A_4)$成立，则存在实常数$\rho>0$，使得边值问题(1)有 2 个解x^*，y^*满足$0<\|x^*\|_E,\|y^*\|_E\le\rho$且$\lim\limits_{m\to\infty}x_m=x^*,\lim\limits_{m\to\infty}y_m=y^*$，其中

$$x_m(t)=Tx_{m-1}(t), y_m(t)=Ty_{m-1}(t),\quad m=1,2,\cdots, \tag{14}$$

$$x_0(t)=\frac{\rho}{\Gamma_q(\alpha)}t^{\alpha-1},\ y_0(t)=0\ , \tag{15}$$

且

$$y_0(t)\le y_1(t)\le\cdots\le y_m(t)\le\cdots\le y^*(t)\le\cdots\le x^*(t)\le\cdots\le x_m(t)\le\cdots\le x_1(t)\le x_0(t),\tag{16}$$

$$D_q^{\gamma_i}y_0(t)\le\cdots\le D_q^{\gamma_i}y_m(t)\le\cdots\le D_q^{\gamma_i}y^*(t)\le\cdots\le D_q^{\gamma_i}x^*(t)\le\cdots\le D_q^{\gamma_i}x_m(t)\le\cdots\le D_q^{\gamma_i}x_0(t)。\tag{17}$$

证明：首先，定义有界集$U_\rho=\{u(t)\in P:\ \|u(t)\|_E\le\rho\}$，其中$\rho$满足

$$\phi_p\left(\frac{\rho}{r}\right)\ge\sup_{\tau\in J}\left\{F_u(\tau)\,|\,u(\tau)\in U_\rho,\left((Q_1u)(\tau),(Q_2u)(\tau)\right)\in[0,h_1^*\rho]\times[0,h_2^*\rho]\right\},\ r=\zeta_pLK_p\Gamma_q(\alpha)。$$

对于任意的$u(t)\in U_\rho$，类似于式(10)和式(11)，有

$$\begin{aligned}\left\|\frac{Tu(t)}{1+t^{\alpha-1}}\right\|_\infty&\le\frac{\zeta_p}{\Gamma_q(\alpha)-\Lambda}\int_0^{+\infty}\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)F_u(\tau)d_q\tau\right)d_qs\\&\le\zeta_pL\int_0^{+\infty}\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)\phi_p\left(\frac{\rho}{r}\right)d_q\tau\right)d_qs\\&\le\zeta_pLK_p\frac{\rho}{r}\le\rho,\end{aligned}$$

且

$$\begin{aligned}\left\|\frac{D_q^{\gamma_i}Tu(t)}{1+t^{\alpha-\gamma_i-1}}\right\|_\infty&\le\frac{\zeta_p\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma_i)\left(\Gamma_q(\alpha)-\Lambda\right)}\int_0^{+\infty}\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)F_u(\tau)d_q\tau\right)d_qs\\&\le\zeta_pL\int_0^{+\infty}\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)\phi_p\left(\frac{\rho}{r}\right)d_q\tau\right)d_qs\\&\le\zeta_pLK_p\frac{\rho}{r}\le\rho,\quad i=1,\cdots,n-1,\end{aligned}$$

所以$\|Tu(t)\|_E\le\rho$，即$TU_\rho\subset U_\rho$。

对于函数列$\{x_m(t)\}_{m=0}^\infty$，不难看出$x_0(t)\in U_\rho$，且由引理 4 得

$$\begin{aligned}x_1(t)=Tx_0(t)&=\zeta_p\int_0^{+\infty}G(t,qs)\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)F_{x_0}(\tau)d_q\tau\right)d_qs\\&\le\frac{\zeta_pt^{\alpha-1}}{\Gamma_q(\alpha)-\Lambda}\int_0^{+\infty}\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)\phi_p\left(\frac{\rho}{r}\right)d_q\tau\right)d_qs\\&\le\zeta_pLK_p\frac{\rho}{r}t^{\alpha-1}=\frac{\rho}{\Gamma_q(\alpha)}t^{\alpha-1}=x_0(t),\end{aligned}$$

当$i=1,\cdots,n-1$时，有

$$\begin{aligned}D_q^{\gamma_i}x_1(t)=D_q^{\gamma_i}Tx_0(t)&=\zeta_p\int_0^{+\infty}G_i^*(t,qs)\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)F_{x_0}(\tau)d_q\tau\right)d_qs\\&\le\frac{\zeta_p\Gamma_q(\alpha)t^{\alpha-\gamma_i-1}}{\Gamma_q(\alpha-\gamma_i)\left(\Gamma_q(\alpha)-\Lambda\right)}\int_0^{+\infty}\phi_p^{-1}\left(\int_0^s(s-q\tau)^{(\beta-1)}a(\tau)\phi_p\left(\frac{\rho}{r}\right)d_q\tau\right)d_qs\\&\le\zeta_pLK_p\frac{\rho}{r}D_q^{\gamma_i}t^{\alpha-1}=\frac{\rho}{\Gamma_q(\alpha)}D_q^{\gamma_i}t^{\alpha-1}=D_q^{\gamma_i}x_0(t)。\end{aligned}$$

又由假设(A_4)中函数f的单调性可知

$$x_2(t)=Tx_1(t)\le Tx_0(t)=x_1(t),$$

$$D_q^{\gamma_i}x_2(t)=D_q^{\gamma_i}Tx_1(t)\le D_q^{\gamma_i}Tx_0(t)=D_q^{\gamma_i}x_1(t),$$

通过归纳，有

$$x_m(t)=Tx_{m-1}(t)\in U_\rho,$$

$$x_m(t)\le x_{m-1}(t),\ D_q^{\gamma_i}x_m(t)\le D_q^{\gamma_i}x_{m-1}(t),\ i=1,\cdots,n-1。$$

对函数列$\{y_m(t)\}_{m=0}^\infty$作类似讨论，易得

$$y_m(t)=Ty_{m-1}(t)\in U_\rho,$$

$$y_m(t)\ge y_{m-1}(t),\ D_q^{\gamma_i}y_m(t)\ge D_q^{\gamma_i}y_{m-1}(t),\ i=1,\cdots,n-1。$$

由 $x_m(t)=Tx_{m-1}(t)\in U_\rho$，$y_m(t)=Ty_{m-1}(t)\in U_\rho$ 及算子 T 的全连续性可知，存在 x^*，$y^*\in U_\rho$ 使得 $\lim\limits_{m\to\infty}x_m=x^*$，$\lim\limits_{m\to\infty}y_m=y^*$ 且 $x^*=Tx^*$，$y^*=Ty^*$。换句话说，x^* 和 y^* 是算子 T 的 2 个不动点。

最后，设 $w(t)$ 是边值问题(1)的任意一个正解，且满足

$$y_0(t)=0\le w(t)\le\frac{\rho}{\Gamma_q(\alpha)}t^{\alpha-1}=x_0(t),$$

$$D_q^{\gamma_i}y_0(t)\le D_q^{\gamma_i}w(t)\le D_q^{\gamma_i}x_0(t),$$

由函数 f 的单调性，进一步有

$$y_1(t)=Ty_0(t)\le Tw(t)=w(t)\le Tx_0(t)=x_1(t),$$

$$D_q^{\gamma_i}y_1(t)=D_q^{\gamma_i}Ty_0(t)\le D_q^{\gamma_i}Tw(t)=D_q^{\gamma_i}w(t)\le D_q^{\gamma_i}Tx_0(t)=D_q^{\gamma_i}x_1(t)\text{。}$$

重复上述迭代过程，则

$$y_m(t)\le w(t)\le x_m(t),\tag{18}$$

$$D_q^{\gamma_i}y_m(t)\le D_q^{\gamma_i}w(t)\le D_q^{\gamma_i}x_m(t)\text{。}\tag{19}$$

对式(18)和式(19)两边同时取极限 $m\to\infty$，则式(16)和式(17)成立。另一方面，由于 $\forall t\in J$，$a(t)f(t,0,\cdots,0)\not\equiv 0$，所以 x^* 和 y^* 分别是边值问题(1)的最大和最小正解。证毕。

3 应用举例

考虑下面分数阶 $q-$差分方程边值问题

$$\begin{cases}D_{0.5}\left(D_{0.5}^{1.8}u(t)\right)+a(t)f\left(t,u(t),\left(Q_1u\right)(t),\left(Q_2u\right)(t),D_{0.5}^{0.5}u(t)\right)=0,\quad t\in[0,+\infty),\\ u(0)=D_{0.5}^{1.8}u(0)=D_{0.5}\left(D_{0.5}^{1.8}u(0)\right)=0,\quad D_{0.5}^{0.8}u(+\infty)=\int_0^{+\infty}e^{-t}u(t)d_qt,\end{cases}\tag{20}$$

其中，$q=0.5$，$p=2$，$\alpha=1.8$，$\beta=1$，$\gamma_1=0.5$，$g(t)=e^{-t}$，

$$f\left(t,u(t),\left(Q_1u\right)(t),\left(Q_2u\right)(t),D_{0.5}^{0.5}u(t)\right)$$

$$=\sin\frac{\pi}{2+t}+\frac{u(t)}{1+t^{0.8}}+\int_0^t\frac{u(t)}{(5+t+s)^2(1+s^{0.8})}d_qs+\int_0^{+\infty}\frac{e^{-4s+t}}{1+s^{0.8}}u(t)d_qs+\frac{D_{0.5}^{0.5}u(t)}{1+t^{0.3}},$$

且

$$H_1(t,s)=\frac{1}{(5+t+s)^2(1+s^{0.8})},\quad H_2(t,s)=\frac{e^{-4s+t}}{1+s^{0.8}}\text{。}$$

显然，$\forall t\in J$，$f(t,0,0,0,0)\not\equiv 0$，且函数 f 关于其每个变量都是单调递增的。同时，通过计算可得 $\Gamma_{0.5}(1.8)\approx 0.9542$，$\Lambda=\int_0^{+\infty}e^{-t}t^{0.8}d_qt\approx 0.6719$，$L\approx 3.6457$，$\zeta_2=1$，且有

$$h_1^*=\sup_{t\in J}\int_0^t\frac{1+s^{0.8}}{(5+t+s)^2(1+s^{0.8})}d_qs=\sup_{t\in J}\int_0^t\frac{1}{(5+t+s)^2}d_qs\le\int_0^{+\infty}\frac{1}{(5+s)^2}d_qs\approx 0.1402,$$

$$h_2^*=\sup_{t\in J}\int_0^{+\infty}\frac{1+s^{0.8}}{e^{4s-t}(1+s^{0.8})}d_qs=\sup_{t\in J}\int_0^{+\infty}e^{-4s+t}d_qs\le\int_0^{+\infty}e^{-4s}d_qs\approx 0.1803\text{。}$$

取 $K_2=0.05$，$\rho=1$，则 $r\approx 0.1739$，且当 $(t,u_1,u_2,u_3,u_4)\in J\times[0,\rho]\times[0,h_1^*\rho]\times[0,h_2^*\rho]\times[0,\rho]$ 时，有

$$f\left(t,(1+t^{0.8})u_1,u_2,u_3,(1+t^{0.3})u_4\right)\le 1+(2+h_1^*+h_2^*)\rho\le 3.3205<\frac{\rho}{r}\approx 5.7504,$$

于是应用定理 1 可知，边值问题(20)有 2 个正解 x^*，y^* 满足 $0<\left\|x^*\right\|_E,\left\|y^*\right\|_E\le 1$。

4 结 语

本文运用单调迭代技巧研究了一类无穷区间上带 $p-$Laplacian 算子的分数阶 $q-$差分方程边值问题正解的存在性。该边值问题的非线性项依赖于未知函数的积分算子及其任意低阶的分数阶 $q-$导数，且当 $q\to 1^-$ 时，边值问题(1)可转化为经典的分数阶微分方程。这使得所得结果更具有一般性，在一定程度上拓展了无穷区间上分数阶 $q-$差分方程非局部边值问题的可解性理论。

但是，本文仅考虑了无穷区间上带 $p-$Laplacian 算子的分数阶 $q-$差分方程解的存在性，对于唯一性和稳定性的研究在现有条件下仍是困难的。因此，今后将利用数值计算、变分迭代、临界点理论等不同方法，

继续深入探讨无穷区间上带 $p-$Laplacian 算子的分数阶 $q-$差分方程非局部问题的可解性和稳定性等基本性质，以揭示其内在的数学规律和实际应用的潜力。

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