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Boros-Moll 多项式序列递推关系的代数证明

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摘　要:为了拓展 Boros-Moll 多项式序列递推关系的基本理论,研究了 Boros-Moll 多项式序列递推关系新的证明方法。首先,对 Boros-Moll 多项式序列满足的递推关系进行适当变形、分析;其次,将满足的递推关系式构造为 3 个部分和的差式;最后,运用代数方法、构造法等数学方法得出 3 个部分的和均为零,进一步得到 Boros-Moll 多项式序列递推关系的一个新的证明方法。结果表明,在 Boros-Moll 多项式序列递推关系中,对其结构进行巧妙变形、分析,再证明相应的引理成立,可得出一个新的证明方法。研究结果丰富了 Boros-Moll 多项式序列递推关系的相关理论,为 Boros-Moll 多项式序列在组合数学、社会科学、信息论等领域的应用提供了理论参考。

关键词:组合数学;Boros-Moll 多项式序列;递推关系;代数证明;构造法

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Algebraic proof of recursive relation for Boros-Moll polynomial sequence

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Abstract: In order to expand the basic theory of the recurrence relationship of Boros-Moll polynomial sequence, a new proof method for the recurrence relationship of Boros-Moll polynomial sequence was studied. Firstly, the recurrence relationship satisfied by the Boros-Moll polynomial sequence was appropriately deformed and partitioned. Secondly, the recursive relationship that satisfies as the difference of the sum of three parts was constructed. Finally, mathematical methods such as algebraic method and structured approach were used to find that the sum of the three parts is all zero. Furthermore, a new proof method for the recurrence relationship of Boros-Moll polynomial sequence was obtained. The results indicate that in the Boros-Moll polynomial sequence recurrence relationship, the recurrence relationship is cleverly deformed and partitioned, and the corresponding lemma is proved to be corrected, thus obtaining a new proof method. The research results enrich the relevant theory of recurrence relationship of the Boros-Moll polynomial sequence, and provide a certain theoretical reference value for the application of the Boros-Moll polynomial sequence in combinatorics, social science, information theory and other fields.

Keywords: combinatorics; Boros-Moll polynomial sequence; recursive relation; algebraic proof; structured approach

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Boros-Moll 多项式序列是在研究一个四次积分求值时[1-2]产生的：

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}\mathrm{d}x=\frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}}P_m(a)\ ,$$

其中，

$$P_m(a)=2^{-2m}\sum_k 2^k\binom{2m-2k}{m-k}\binom{m+k}{k}(a+1)^k,$$

称为 Boros-Moll 多项式。令 $d_i(m)$是 $P_m(a)$中 a^i 项的系数，当 $0\leqslant i\leqslant m$ 时，有

$$P_m(a)=\sum_{i=0}^m d_i(m)a^i,$$

其中，

$$d_i(m)=2^{-2m}\sum_{k=i}^m 2^k\binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{i},$$

称为 Boros-Moll 序列。

BOROS 等[3]证明了当 $m\geqslant 0$ 时，序列$\{d_i(m)\}_{0\leqslant i\leqslant m}$ 具有单峰性；MOLL[4]猜想当 $m\geqslant 2$ 时，这个序列具有对数凹性，即 $d_i(m)^2\geqslant d_{i-1}(m)d_{i+1}(m)(1\leqslant i\leqslant m-1)$；之后，国内外对 Boros-Moll 序列性质的研究取得了大量成果，参见文献[5]—文献[14]。LIU[15]给出了 Boros-Moll 序列的斜对数凹性；CHEN 等[16]证明了 Boros-Moll 序列的逆超对数凹性，证明了$\{d_i(m)\}_{0\leqslant i\leqslant m}$ 的 2-对数凹性[17]，给出了 Boros-Moll 多项式序列正性的组合证明[18]；HAN 等[19]研究了 Boros-Moll 多项式的对称分解，证明了 Boros-Moll 多项式的交替双 γ-正性。国外关于 Boros-Moll 多项式序列问题的相关研究参见文献[20]—文献[21]。

2007 年，KAUERS 等[22]用机器证明的方法给出了 Boros-Moll 序列满足递推关系，进而证实了 MOLL[4]猜想。机器证明可以快速地用计算机自动推理生成定理的证明方法，但缺少严格的数学证明，本文给出定理的严格数学证明。

定理 1[22]　当 $m\geqslant 2,0\leqslant i\leqslant m+1$ 时，有

$$-(-2+i-m)(-1+i+m)d_{i-2}(m)-(i-1)(2m+1)d_{i-1}(m)+i(i-1)d_i(m)=0\ 。\tag{1}$$

基于代数方法，本文给出定理 1 的代数证明。

1　对递推关系的变形、分拆

为了证明式(1)的递推关系，对该式作适当变形、分拆，构造出 3 个部分和为零的差式，进而证明定理。PANG 等[23]在证明 Boros-Moll 系数序列最大下界的过程中，得到如下结论：

$$D_i(m)=\binom{2m}{m-i}m!\ i!\ (m-i)!\ 2^m d_i(m)\ ,\tag{2}$$

$$D_{i,j}(m)=\binom{2m}{m-i}\binom{m-i}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)!\ \ ,\tag{3}$$

式中：$(x)_n$ 表示上升的阶乘，当 $n=0$ 时，$(x)_0=1$；当 $n>0$ 时，$(x)_n=x(x+1)\cdots(x+n-1)$，
因此，有

$$D_i(m)=\sum_{j=0}^{m-i}D_{i,j}(m)\ 。\tag{4}$$

利用上述知识将定理 1 变形、分拆，得到 3 个引理。

引理 1

$$(m+i-1)D_{i-1}(m)=\sum_{j=0}^{m-i+2}(m-i-j+2)D_{i-2,j}(m)\ 。\tag{5}$$

证明：利用式(3)和式(4)，有

$$(m+i-1)D_{i-1}(m)=(m+i-1)\sum_{j=0}^{m-i+1}D_{i-1,j}(m)=$$

$$(m+i-1)\sum_{j=0}^{m-i+1}\binom{2m}{m-i+1}\binom{m-i+1}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)!\ =$$

$$(m+i-1)\sum_{j=0}^{m-i+1}\frac{(2m)!}{(m-i+1)!\,(m+i-1)!}\cdot\frac{(m-i+1)!}{j!\,(m-i-j+1)!}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)! =$$

$$\sum_{j=0}^{m-i+1}\frac{(2m)!}{(m+i-2)!\,j!\,(m-i-j+1)!}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)! =$$

$$\sum_{j=0}^{m-i+1}\frac{(2m)!}{(m+i-2)!\,(m-i+2)!}\cdot\frac{(m-i+2)!}{j!\,(m-i-j+1)!}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)! =$$

$$\sum_{j=0}^{m-i+1}(m-i-j+2)\frac{(2m)!}{(m+i-2)!\,(m-i+2)!}\cdot\frac{(m-i+2)!}{j!\,(m-i-j+2)!}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)! =$$

$$\sum_{j=0}^{m-i+1}(m-i-j+2)\binom{2m}{m-i+2}\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)! =$$

$$\sum_{j=0}^{m-i+1}(m-i-j+2)D_{i-2,j}(m)=$$

$$\sum_{j=0}^{m-i+2}(m-i-j+2)D_{i-2,j}(m)。$$

证毕。

引理 2

$$\sum_{j=0}^{m-i+2}jD_{i-2,j}(m)=\sum_{j=0}^{m-i+1}(j+1)D_{i-1,j}(m)\ 。\tag{6}$$

证明：根据式(3)，能够得到：

$$\sum_{j=0}^{m-i+2}jD_{i-2,j}(m)=$$

$$\sum_{j=0}^{m-i+2}j\binom{2m}{m-i+2}\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)! =$$

$$\binom{2m}{m-i+2}\sum_{j=0}^{m-i+2}j\cdot\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)! =$$

$$\frac{(2m)!}{(m-i+2)!\,(m+i-2)!}\sum_{j=0}^{m-i+2}j\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)! =$$

$$\frac{(2m)!}{(m-i+1)!\,(m+i-1)!}\cdot\frac{m+i-1}{m-i+2}\sum_{j=0}^{m-i+2}j\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)! =$$

$$\binom{2m}{m-i+1}\cdot\frac{m+i-1}{m-i+2}\sum_{j=0}^{m-i+2}j\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)! =$$

$$\binom{2m}{m-i+1}\cdot\frac{1}{m-i+2}\sum_{j=0}^{m-i+2}(m+i-1)\cdot j\cdot\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)! =$$

$$\binom{2m}{m-i+1}\cdot\frac{1}{m-i+2}\sum_{j=1}^{m-i+2}(m+i-1)\cdot j\cdot\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)! =$$

$$\binom{2m}{m-i+1}\cdot\frac{1}{m-i+2}\sum_{j=1}^{m-i+2}[(2m-j+1)-(m-i-j+2)]\cdot j\cdot\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)! =$$

$$\binom{2m}{m-i+1}\cdot\frac{1}{m-i+2}\cdot\left[\sum_{j=1}^{m-i+2}(2m-j+1)\cdot j\cdot\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)! -\right.$$

$$\left.\sum_{j=1}^{m-i+2}(m-i-j+2)\cdot j\cdot\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)!\ \right]=$$

$$\binom{2m}{m-i+1}\cdot\frac{1}{m-i+2}\cdot\left[\sum_{j=1}^{m-i+2}(2m-j+1)\cdot j\cdot\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)! -\right.$$

$$\left.\sum_{j=1}^{m-i+1}(m-i-j+2)\cdot j\cdot\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)!\ \right]=$$

$$\binom{2m}{m-i+1}\cdot\frac{1}{m-i+2}\cdot\left[\sum_{j=1}^{m-i+2}j\cdot\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j+1)! -\right.$$

$$\sum_{j=1}^{m-i+1}(m-i-j+2)\cdot j\cdot\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)!\Bigg]=$$

$$\binom{2m}{m-i+1}\cdot\frac{1}{m-i+2}\cdot\Bigg[\sum_{k=0}^{m-i+1}(k+1)\cdot\binom{m-i+2}{k+1}\cdot 2^{k+1}\cdot\left(\frac{1}{2}\right)_{k+1}\cdot(2m-k)!-$$

$$\sum_{j=1}^{m-i+1}(m-i-j+2)\cdot j\cdot\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)!\Bigg]=$$

$$\binom{2m}{m-i+1}\cdot\frac{1}{m-i+2}\cdot\Bigg[\sum_{j=0}^{m-i+1}(j+1)\cdot\binom{m-i+2}{j+1}\cdot 2^{j+1}\cdot\left(\frac{1}{2}\right)_{j+1}\cdot(2m-j)!-$$

$$\sum_{j=1}^{m-i+1}(m-i-j+2)\cdot j\cdot\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)!\Bigg]=$$

$$\binom{2m}{m-i+1}\cdot\frac{1}{m-i+2}\cdot\Bigg[(m-i+2)\cdot(2m)!+\sum_{j=1}^{m-i+1}(j+1)\cdot\binom{m-i+2}{j+1}\cdot$$

$$2^{j+1}\cdot\left(\frac{1}{2}\right)_{j+1}\cdot(2m-j)!-\sum_{j=1}^{m-i+1}(m-i-j+2)\cdot j\cdot\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)!\Bigg]=$$

$$\binom{2m}{m-i+1}\cdot\frac{1}{m-i+2}\cdot\Bigg\{(m-i+2)\cdot(2m)!+\sum_{j=1}^{m-i+1}\Bigg[(j+1)\cdot\binom{m-i+2}{j+1}\cdot 2^{j+1}\cdot\left(\frac{1}{2}\right)_{j+1}-$$

$$(m-i-j+2)\cdot j\cdot\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\Bigg]\cdot(2m-j)!\Bigg\}=$$

$$\binom{2m}{m-i+1}\cdot\Bigg\{(2m)!+\frac{1}{m-i+2}\sum_{j=1}^{m-i+1}\Bigg[(j+1)\cdot\binom{m-i+2}{j+1}\cdot 2^{j+1}\cdot\left(\frac{1}{2}\right)_{j+1}-$$

$$(m-i-j+2)\cdot j\cdot\binom{m-i+2}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\Bigg]\cdot(2m-j)!\Bigg\}=$$

$$\binom{2m}{m-i+1}\cdot\Bigg\{(2m)!+\frac{1}{m-i+2}\sum_{j=1}^{m-i+1}\Bigg[(j+1)\cdot\frac{(m-i+2)!}{(j+1)!\,(m-i-j+1)!}\cdot 2^j\cdot$$

$$2\cdot\left(\frac{1}{2}\right)_j\cdot\left(\frac{1}{2}+j\right)-(m-i-j+2)\cdot j\cdot\frac{(m-i+2)!}{j!\,(m-i-j+2)!}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\Bigg]\cdot(2m-j)!\Bigg\}=$$

$$\binom{2m}{m-i+1}\cdot\Bigg\{(2m)!+\sum_{j=1}^{m-i+1}\Bigg[(2j+1)\cdot\binom{m-i+1}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j-j\cdot\binom{m-i+1}{j}\cdot$$

$$2^j\cdot\left(\frac{1}{2}\right)_j\Bigg]\cdot(2m-j)!\Bigg\}=$$

$$\binom{2m}{m-i+1}\cdot\Bigg[(2m)!+\sum_{j=1}^{m-i+1}(j+1)\cdot\binom{m-i+1}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)!\Bigg]=$$

$$\binom{2m}{m-i+1}\cdot\sum_{j=0}^{m-i+1}(j+1)\cdot\binom{m-i+1}{j}\cdot 2^j\cdot\left(\frac{1}{2}\right)_j\cdot(2m-j)!=$$

$$\sum_{j=0}^{m-i+1}(j+1)D_{i-1,j}(m)\ 。$$

证毕。

引理 3

$$(m+i)D_i(m)=\sum_{j=0}^{m-i+1}(m-i-j+1)D_{i-1,j}(m)。\tag{7}$$

证明：显然，将式(5)中的 $i-1$ 替换为 i，即可得到式(7)，证毕。

2　对定理 1 的证明

为了证明递推关系式(1)，建立用 $D_i(m)$ 表示的等价形式。利用式(2)递推式，式(1)可以表示为

$$(m+i)D_i(m)+(m-i+2)D_{i-2}(m)-(2m+1)D_{i-1}(m)=0\ ,\tag{8}$$

因此，定理 1 证明如下。

证明：根据式(4)，有：

$$\begin{aligned}&(m+i)D_i(m)+(m-i+2)D_{i-2}(m)-(2m+1)D_{i-1}(m)=\\&\quad(m+i)D_i(m)+(m-i+2)\sum_{j=0}^{m-i+2}D_{i-2,j}(m)-(2m+1)\sum_{j=0}^{m-i+1}D_{i-1,j}(m)=\\&\quad(m+i)D_i(m)+\Big[\sum_{j=0}^{m-i+2}jD_{i-2,j}(m)+\sum_{j=0}^{m-i+2}(m-i-j+2)D_{i-2,j}(m)\Big]-\\&\quad\Big[\sum_{j=0}^{m-i+1}(m-i-j+1)D_{i-1,j}(m)+\sum_{j=0}^{m-i+1}(j+1)D_{i-1,j}(m)+\sum_{j=0}^{m-i+1}(m+i-1)D_{i-1,j}(m)\Big]=\\&\quad\Big[\sum_{j=0}^{m-i+2}(m-i-j+2)D_{i-2,j}(m)-\sum_{j=0}^{m-i+1}(m+i-1)D_{i-1,j}(m)\Big]+\\&\quad\Big[\sum_{j=0}^{m-i+2}jD_{i-2,j}(m)-\sum_{j=0}^{m-i+1}(j+1)D_{i-1,j}(m)\Big]+\\&\quad\Big[(m+i)D_i(m)-\sum_{j=0}^{m-i+1}(m-i-j+1)D_{i-1,j}(m)\Big]=\\&\quad\Big[\sum_{j=0}^{m-i+2}(m-i-j+2)D_{i-2,j}(m)-(m+i-1)D_{i-1}(m)\Big]+\\&\quad\Big[\sum_{j=0}^{m-i+2}jD_{i-2,j}(m)-\sum_{j=0}^{m-i+1}(j+1)D_{i-1,j}(m)\Big]+\\&\quad\Big[(m+i)D_i(m)-\sum_{j=0}^{m-i+1}(m-i-j+1)D_{i-1,j}(m)\Big]。\end{aligned}$$

由式(5)—式(7)，可得：

$$\begin{aligned}&\Big[\sum_{j=0}^{m-i+2}(m-i-j+2)D_{i-2,j}(m)-(m+i-1)D_{i-1}(m)\Big]+\\&\quad\Big[\sum_{j=0}^{m-i+2}jD_{i-2,j}(m)-\sum_{j=0}^{m-i+1}(j+1)D_{i-1,j}(m)\Big]+\\&\quad\Big[(m+i)D_i(m)-\sum_{j=0}^{m-i+1}(m-i-j+1)D_{i-1,j}(m)\Big]=0。\end{aligned}$$

证毕。

3 结 语

1)本文基于 KAUERS 等研究 Boros-Moll 多项式序列对数凹性时提出的一个递推关系，研究了 Boros-Moll 多项式序列递推关系的代数证明方法。通过对其递推关系进行巧妙变形、分拆，证明了相应的引理，进而给出了代数证明方法。

2)所得结果在一定程度上丰富了 Boros-Moll 多项式序列递推关系的相关理论，为 Boros-Moll 多项式序列在组合数学、社会科学、信息论等领域的应用提供了一定的理论参考价值。

鉴于证明方法的多样性，本研究方法主要考虑了 Boros-Moll 多项式递推关系的代数证明，将递推关系式写成了 3 个式子的和。未来可以考虑更加简便、直观的研究方法，如组合分析法、赋权组合结构法，赋予 3 个式子组合解释，间接给出递推关系的组合证明，增加递推关系运用的灵活性。

参考文献/References：

[1] BOROS G，MOLL V H. An integral hidden in Gradshteyn and Ryzhik[J]. Journal of Computational and Applied Mathematics，1999，106(2)：361-368.

[2] BOROS G，MOLL V H. The double square root，Jacobi polynomials and Ramanujan's master theorem[J]. Journal of Computational and Applied Mathematics，2001，130(1/2)：337-344.

[3] BOROS G，MOLL V H. A criterion for unimodality[J]. The Electronic Journal of Combinatorics，1999，6：10-11.

[4] MOLL V H. The evaluation of integrals：A personal story[J]. Notices of the American Mathematical Society，2002，49(3)：311-317.

[5] AMDEBERHAN T，MANNA D V，MOLL V H. The 2-adic valuation of a sequence arising from a rational integral[J]. Journal of Combinatorial Theory，Series A，2008，115(8)：1474-1486.

[6] AMDEBERHAN T,MOLL V H. A formula for a quartic integral:A survey of old proofs and some new ones[J]. The Ramanujan Journal,2009,18(1):91-102.

[7] CHEN W Y C,DOU D Q J,YANG A L B. Brändén's conjectures on the Boros-Moll polynomials[J]. International Mathematics Research Notices,2013(20):4819-4828.

[8] CHEN W Y C,WANG L X W,XIA E X W. Interlacing log-concavity of the Boros-Moll polynomials[J]. Pacific Journal of Mathematics,2011,254(1):89-99.

[9] LYU L. A short proof of Moll's minimal conjecture[J]. The Electronic Journal of Combinatorics,2017,24(4): 4-7.

[10] MOLL V H. Combinatorial sequences arising from a rational integral[J]. Online Journal of Analytic Combinatorics,2007,2(2):456-473.

[11] MOLL V H,MANNA D V. A remarkable sequence of integers[J]. Expositiones Mathematicae,2009,27(4):289-312.

[12] SUN Xinyu,MOLL V H. A binary tree representation for the 2-adic valuation of a sequence arising from a rational integral[J]. Integers,2010,10(2):211-222.

[13] XIA E X W. The concavity and convexity of the Boros-Moll sequences[J]. The Electronic Journal of Combinatorics,2015,22(1):1-8.

[14] CHEN W Y C,XIA E X W. The ratio monotonicity of the Boros-Moll polynomials[J]. Mathematics of Computation,2009,78(268):2269-2282.

[15] LIU E H. Skew log-concavity of the Boros-Moll sequences[J]. Journal of Inequalities and Applications,2017(1):117-121.

[16] CHEN W Y C,GU C Y. The reverse ultra log-concavity of the Boros-Moll polynomials[J]. Proceedings of the American Mathematical Society,2009,137(12):3991-3998.

[17] CHEN W Y C,XIA E X W. 2-log-concavity of the Boros-Moll polynomials[J]. Proceedings of the Edinburgh Mathematical Society,2013,56(3):701-722.

[18] CHEN W Y C,PANG S X M,QU E X Y. On the combinatorics of the Boros-Moll polynomials[J]. The Ramanujan Journal,2010,21(1):41-51.

[19] HAN Guoniu,MA Shimei,YEH Y N. A Symmetric Decomposition of the Boros-Moll Polynomials[DB/OL]. https://arxiv. org/abs/2212. 14501,2022-12-30.

[20] SAGAN B E. Log concave sequences of symmetric functions and analogs of the Jacobi-Trudi determinants[J]. Transactions of the American Mathematical Society,1992,329(2):795-811.

[21] MCNAMARA P R W,SAGAN B E. Infinitelog-concavity:Developments and conjectures[J]. Advances in Applied Mathematics,2010,44(1):1-15.

[22] KAUERS M,PAULE P. A computer proof of Moll's log-concavity conjecture[J]. Proceedings of the American Mathematical Society,2007,135(12):3847-3856.

[23] PANG S X M,LYU L,WANG J X. The greatest lower bound of a Boros-Moll sequence[J]. Mathematical Notes,2022,111(1):115-123.