文章编号:1008-1542(2023)02-0132-12

具有 Hilfer 分数阶脉冲微分方程边值问题解的存在性

郭春静[1],孟凡猛[1],陈 坤[2],江卫华[1]

(1.河北科技大学理学院,河北石家庄 050018;2.石家庄人民医学高等专科学校教务处,河北石家庄 050091)

摘 要:为了拓展边值问题的基本理论,研究一类具有有限个脉冲点的 Hilfer 分数阶脉冲微分方程边值问题解的存在性。首先,求出微分方程等价的积分方程;其次,定义恰当的 Banach 空间和范数,构造合适的算子,在非线性项满足不同条件的情况下,运用 Krasnoselskii 不动点定理,分别得到此类边值问题存在解的充分条件;最后,通过 2 个实例验证研究结果的普适性。结果表明,含有 Hilfer 分数阶导数的脉冲微分方程边值问题的解具有存在性。运用 Krasnoselskii 不动点定理能够有效解决具有 Hilfer 分数阶脉冲微分方程边值问题解的存在性问题,丰富了分数阶微分方程理论,为解决其他类型的脉冲分数阶微分方程边值问题提供了借鉴与参考。

关键词:解析理论;脉冲;边值问题;Krasnoselskii 不动点定理;解的存在性

中图分类号:O175.8 **文献标识码**:A **DOI**:10.7535/hbkd.2023yx02004

Existence of solutions for boundary value problems of fractional impulsive differential equations with Hilfer

GUO Chunjing[1], MENG Fanmeng[1], CHEN Kun[2], JIANG Weihua[1]

(1. School of Sciences, Hebei University of Science and Technology, Shijiazhuang, Hebei 050018, China; 2. Office of Academic Affairs, Shijiazhuang People's Medical College, Shijiazhuang, Hebei 050091, China)

Abstract: In order to extend the basic theory of boundary value problems, the existence of solutions for a class of Hilfer fractional impulsive differential equations with finite impulsive points was studied. Firstly, the integral equation equivalent to the differential equation was obtained; Secondly, appropriate Banach spaces and norms were defined, and appropriate operators were constructed. When the nonlinear term satisfies different conditions, sufficient conditions for the existence of solutions of such boundary value problems were obtained by using Krasnoselskii fixed point theorem; Finally, two examples were used to

收稿日期:2022-11-11;修回日期:2023-01-13;责任编辑:张士莹

基金项目:国家自然科学基金(11775169);河北省自然科学基金(A2018208171)

第一作者简介:郭春静(1996—),女,河北沧州人,硕士研究生,主要从事应用泛函分析、微分方程边值问题方面的研究。

通信作者:江卫华教授。E-mail:jianghua64@163.com

郭春静,孟凡猛,陈坤,等.具有 Hilfer 分数阶脉冲微分方程边值问题解的存在性[J].河北科技大学学报,2023,44(2):132-143.

GUO Chunjing, MENG Fanmeng, CHEN Kun, et al. Existence of solutions for boundary value problems of fractional impulsive differential equations with Hilfer[J]. Journal of Hebei University of Science and Technology, 2023, 44(2): 132-143.

illustrate the universality of the research results. It is shown that the solution of the boundary value problem of impulsive differential equations with Hilfer fractional derivative exists. By using the Krasnoselskii fixed-point theorem, the existence of solutions for impulsive differential equation boundary value problems with Hilfer fractional order can be effectively solved, which provides some reference for solving other types of impulsive fractional differential equation boundary value problems.

Keywords: analytic theory; impulse; boundary value problem; Krasnoselskii fixed point theorem; existence of solutions

近几十年来，分数阶微分方程受到研究者的广泛关注。人们之所以对分数阶微分方程产生兴趣，主要是因为分数阶导数对于科学技术领域的不同过程、材料记忆以及遗传特性描述发挥着重要作用。脉冲现象实际上是一种间断、突然的变化，经常伴随一些物理系统的出现。脉冲微分方程广泛应用于力学、医学、生态学等诸多领域。为了更加精确地描述这类演化过程，许多研究人员对具有脉冲条件的微分方程展开讨论，参见文献[1]—文献[9]。在文献[10]中，FENG 等运用不动点定理研究了下列整数阶脉冲微分方程边值问题：

$$\begin{cases} -x''(t)=f(t,x(t)), & t\in J, t\neq t_k, \\ -\Delta x'|_{t=t_k}=I_k(x(t_k)), & k=1,2,\cdots,n, \\ x(0)=\sum_{i=1}^{m-2}a_i x(\xi_i), & x(1)=\sum_{i=1}^{m-2}b_i x(\xi_i), \end{cases}$$

其中 $J=[0,1]$，$f\in C(J\times R^+,R^+)$，$I_k\in C(R^+,R^+)$，$R^+=[0,+\infty)$，得到了其正解的存在性结果。由于整数阶导数推广到了分数阶导数，所以人们的研究工作也从整数阶脉冲微分方程拓展到了分数阶脉冲微分方程，参见文献[11]—文献[15]。江卫华等[16]利用 Banach 压缩映像原理和 Krasnoselskii 不动点定理，探究了具有 Riemann-Liouville 导数的分数阶微分方程边值问题：

$$\begin{cases} D_{0+}^{\alpha}u(t)=f(t,u(t),D_{0+}^{\alpha-1}u(t)), \\ \Delta D_{0+}^{\alpha-1}u|_{t=t_k}=I_k(u(t_k^-)), \quad k=1,2,\cdots, \\ u(0)=0,\ D_{0+}^{\alpha-1}u(\infty)=\int_0^{\infty}h(t)D_{0+}^{\alpha-1}u(t)\mathrm{d}t, \end{cases}$$

得到了此类问题解的存在性，其中 $1<\alpha<2$，D_{0+}^{α} 是 Riemann-Liouville 导数。

近年来，人们对分数阶导数的定义由 Riemann-Liouville 分数阶导数和 Caputo 分数阶导数推广到了 Hilfer 分数阶导数。Hilfer 分数阶导数中含有 2 个参量，即 α 和 β。当 $\beta=0$ 时，Hilfer 分数阶导数即为 Riemann-Liouville 分数阶导数；当 $\beta=1$ 时，Hilfer 分数阶导数即为 Caputo 分数阶导数；当 $0<\beta<1$ 时，Hilfer 分数阶导数介于 2 种导数之间。Hilfer 分数阶导数的性质更具有一般性，具有 Hilfer 分数阶导数的微分方程在解决实际问题方面扮演着越来越重要的角色[17-19]。本文应用 Krasnoselskii 不动点定理，研究具有有限个脉冲点的 Hilfer 分数阶脉冲边值问题

$$\begin{cases} D_{0+}^{\alpha,\beta}x(t)=f(t,x(t),D_{0+}^{\gamma-2}x(t),D_{0+}^{\gamma-1}x(t)),\ t\in J'=J\setminus\{t_1,t_2,\cdots,t_m\}, \\ \Delta x(t_k)=I_k(x(t_k)),\Delta D_{0+}^{\gamma-2}x(t_k)=\bar{I}_k(x(t_k)),\Delta D_{0+}^{\gamma-1}x(t_k)=\tilde{I}_k(x(t_k)), \\ x(0)=0,D_{0+}^{\gamma-2}x(0)=aD_{0+}^{\gamma-1}x(1),\ D_{0+}^{\gamma-2}x(1)=bD_{0+}^{\gamma-1}x(0) \end{cases} \tag{1}$$

解的存在性。其中：$2<\alpha<3$，$0\leqslant\beta\leqslant1$，$\gamma=\alpha+3\beta-\alpha\beta$，$a,b\in R$，并且 $a\geqslant1$，$b>a+1$；$D_{0+}^{\alpha,\beta}$ 是 Hilfer 分数阶导数；$D_{0+}^{\gamma-2}$，$D_{0+}^{\gamma-1}$ 是标准的 Riemann-Liouville 分数阶导数。令 $J=[0,1]$，$J_0=[0,t_1]$，$J_k=(t_k,t_{k+1}]$，$k=1,2,\cdots,m$，$m\in\mathbf{Z}^+$，$0=t_0<t_1<\cdots<t_k<\cdots<t_m<t_{m+1}=1$，$f\in C(J\times R^3,R)$，$I_k,\bar{I}_k,\tilde{I}_k\in C(R,R)$；$\Delta x(t_k)=x(t_k^+)-x(t_k^-)$，$x(t_k^+)=\lim_{h\to0^+}x(t_k+h)$，$x(t_k^-)=\lim_{h\to0^-}x(t_k+h)$。$\Delta D_{0+}^{\gamma-2}x(t_k)=D_{0+}^{\gamma-2}x(t_k^+)-D_{0+}^{\gamma-2}x(t_k^-)$，$\Delta D_{0+}^{\gamma-2}x(t_k^+)=\lim_{h\to0^+}D_{0+}^{\gamma-2}x(t_k+h)$，$\Delta D_{0+}^{\gamma-2}x(t_k^-)=\lim_{h\to0^-}D_{0+}^{\gamma-2}x(t_k+h)$。$\Delta D_{0+}^{\gamma-1}x(t_k)=D_{0+}^{\gamma-1}x(t_k^+)-D_{0+}^{\gamma-1}x(t_k^-)$，$\Delta D_{0+}^{\gamma-1}x(t_k^+)=\lim_{h\to0^+}D_{0+}^{\gamma-1}x(t_k+h)$，$\Delta D_{0+}^{\gamma-1}x(t_k^-)=\lim_{h\to0^-}D_{0+}^{\gamma-1}x(t_k+h)$。

1　预备知识及引理

定义 1[20]　假设 $n-1<\alpha<n$，$0\leqslant\beta\leqslant1$，则 $y\in C^n([0,1],R)$的 α 阶 β 型 Hilfer 分数阶导数的定义为

$$D_{0+}^{\alpha;\beta}\, y(t)=I_{0+}^{\beta(n-\alpha)}\,(\frac{d}{dt})^{n}(I_{0+}^{(1-\beta)(n-\alpha)})y(t)。$$

注：定义1还可表示为 $D_{0+}^{\alpha;\beta}y(t)=I_{0+}^{\beta(n-\alpha)}D_{0+}^{\gamma}y(t)$，其中 $\gamma=\alpha+n\beta-\alpha\beta$，并且易得 $n-1<\alpha\leqslant\gamma\leqslant n$。

引理1[21] $D_{0+}^{\alpha}y(t)=0\Leftrightarrow y(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n}$，其中 $c_i\in R,i=1,2,\cdots,n,n=[\alpha]+1$。

引理2[21] 若 $\alpha>0,\lambda>-1$，则 $D_{0+}^{\alpha}t^{\lambda}=\dfrac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}t^{\lambda-\alpha}$。

引理3[22] 设 Ω 是 Banach 空间 E 中的一个非空凸闭子集，且设 P,Q 是2个算子，满足：a)对任意的 $x,y\in\Omega$，有 $Px+Qy\in\Omega$；b)P 是全连续映射；c)Q 是一个压缩映射，则至少存在一个 $z\in\Omega$，使得 $z=Pz+Qz$。

定义空间 $PC_{\gamma}[J,R]=\{x:x,D_{0+}^{\gamma-2}x,D_{0+}^{\gamma-1}x\in C(J_k),x(t_k^{+}),x(t_k^{-}),D_{0+}^{\gamma-2}x(t_k^{+}),D_{0+}^{\gamma-2}x(t_k^{-}),D_{0+}^{\gamma-1}x(t_k^{+}),D_{0+}^{\gamma-1}x(t_k^{-})$ 存在，$x(t_k^{-})=x(t_k),D_{0+}^{\gamma-2}x(t_k^{-})=D_{0+}^{\gamma-2}x(t_k),D_{0+}^{\gamma-1}x(t_k^{-})=D_{0+}^{\gamma-1}x(t_k),k=1,2,\cdots,m\}$，$2<\gamma\leqslant3$。$PC_{\gamma}^{1}[J,R]=\{x:x\in PC_{\gamma}[J,R]\}$，$2<\gamma\leqslant3$，范数定义为 $\|x\|=\max\{\|x\|_{\infty},\|D_{0+}^{\gamma-2}x\|_{\infty},\|D_{0+}^{\gamma-1}x\|_{\infty}\}$，其中 $\|x\|_{\infty}=\sup\limits_{t\in J}|x(t)|$，$\|D_{0+}^{\gamma-2}x\|_{\infty}=\sup\limits_{t\in J}|D_{0+}^{\gamma-2}x(t)|$，$\|D_{0+}^{\gamma-1}x\|_{\infty}=\sup\limits_{t\in J}|D_{0+}^{\gamma-1}x(t)|$，显然，$PC_{\gamma}^{1}[J,R]$ 是 Banach 空间。

引理4 若 $y(t)\in C[0,1]$，$2<\alpha<3$，$0\leqslant\beta\leqslant1$，$\gamma=\alpha+3\beta-\alpha\beta$，$a\geqslant1$，$b>a+1$，$x(t)$ 是分数阶脉冲微分方程边值问题

$$\begin{cases}D_{0+}^{\alpha;\beta}\,x(t)=y(t),\quad t\in J'=J\backslash\{t_1,t_2,\cdots,t_m\},\\ \Delta x(t_k)=I_k(x(t_k)),\quad \Delta D_{0+}^{\gamma-2}\,x(t_k)=\bar{I}_k(x(t_k)),\quad \Delta D_{0+}^{\gamma-1}\,x(t_k)=\tilde{I}_k(x(t_k))\\ x(0)=0,\quad D_{0+}^{\gamma-2}\,x(0)=aD_{0+}^{\gamma-1}\,x(1),\quad D_{0+}^{\gamma-2}\,x(1)=bD_{0+}^{\gamma-1}\,x(0)\end{cases}\tag{2}$$

的解当且仅当 $x(t)$ 满足积分方程：

$$x(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds+\frac{t^{\gamma-1}+a(\gamma-1)t^{\gamma-2}}{(b-a-1)\Gamma(\gamma)}\int_0^1\frac{(1-s)^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}y(s)ds+$$
$$\frac{a[t^{\gamma-1}+(b-1)(\gamma-1)t^{\gamma-2}]}{(b-a-1)\Gamma(\gamma)}\int_0^1\frac{(1-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}y(s)ds+\frac{a[t^{\gamma-1}+(b-1)(\gamma-1)t^{\gamma-2}]}{(b-a-1)\Gamma(\gamma)}\sum_{k=1}^{m}\tilde{I}_k(x(t_k))+$$
$$\frac{t^{\gamma-1}+a(\gamma-1)t^{\gamma-2}}{(b-a-1)\Gamma(\gamma)}\sum_{k=1}^{m}[\tilde{I}_k(x(t_k))(1-t_k)+\bar{I}_k(x(t_k))]+t^{\gamma-3}\sum_{0<t_k<t}\left\{\frac{\tilde{I}_k(x(t_k))}{\Gamma(\gamma)}(t-t_k)[t-(\gamma-2)t_k]+\right.$$
$$\left.\frac{\bar{I}_k(x(t_k))}{\Gamma(\gamma-1)}(t-t_k)+I_k(x(t_k))t_k^{3-\gamma}\right\}。\tag{3}$$

证明：设 $x(t)$ 是边值问题(2)的解。当 $t\in J_0$ 时，由定义1的注，可得

$$I_{0+}^{\beta(3-\alpha)}\,D_{0+}^{\gamma}\,x(t)=y(t)。\tag{4}$$

对式(4)两边作用 $I_{0+}^{\alpha}(\cdot)$，得 $I_{0+}^{\alpha}I_{0+}^{\beta(3-\alpha)}D_{0+}^{\gamma}x(t)=I_{0+}^{\alpha}y(t)$，即 $I_{0+}^{\gamma}D_{0+}^{\gamma}x(t)=I_{0+}^{\alpha}y(t)$，由引理1得

$$x(t)=I_{0+}^{\alpha}\,y(t)+c_1t^{\gamma-1}+c_2t^{\gamma-2}+c_3t^{\gamma-3},c_i\in R,i=1,2,3。$$

因为 $x(0)=0$，得 $c_3=0$，故 $x(t)=\int_0^t\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds+c_1t^{\gamma-1}+c_2t^{\gamma-2}$。根据引理2，得到

$$D_{0+}^{\gamma-2}\,x(t)=\int_0^t\frac{(t-s)^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}y(s)ds+c_1\Gamma(\gamma)t+c_2\Gamma(\gamma-1),$$
$$D_{0+}^{\gamma-1}\,x(t)=c_1\Gamma(\gamma)+\int_0^t\frac{(t-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}y(s)ds。$$

因此，

$$x(t_1^{-})=\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds+c_1t_1^{\gamma-1}+c_2t_1^{\gamma-2},$$
$$D_{0+}^{\gamma-2}\,x(t_1^{-})=\int_0^{t_1}\frac{(t_1-s)^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}y(s)ds+c_1\Gamma(\gamma)t_1+c_2\Gamma(\gamma-1),$$
$$D_{0+}^{\gamma-1}\,x(t_1^{-})=\int_0^{t_1}\frac{(t_1-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}y(s)ds+c_1\Gamma(\gamma)。$$

当 $t\in J_1$ 时，$x(t)=\int_0^t\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds+d_1t^{\gamma-1}+d_2t^{\gamma-2}+d_3t^{\gamma-3}$，$d_i\in R,i=1,2,3$。

所以，

$$x(t_1^{+})=\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds+d_1t_1^{\gamma-1}+d_2t_1^{\gamma-2}+d_3t_1^{\gamma-3},$$

$$D_{0+}^{\gamma-2}x(t_1^+)=\int_0^{t_1}\frac{(t_1-s)^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}y(s)\mathrm{d}s+d_1\Gamma(\gamma)t_1+d_2\Gamma(\gamma-1),$$

$$D_{0+}^{\gamma-1}x(t_1^+)=\int_0^{t_1}\frac{(t_1-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}y(s)\mathrm{d}s+d_1\Gamma(\gamma)。$$

根据脉冲条件 $\Delta x(t_1)=I_1(x(t_1))$,$\Delta D_{0+}^{\gamma-2}x(t_1)=\bar{I}_1(x(t_1))$,$\Delta D_{0+}^{\gamma-1}x(t_1)=\tilde{I}_1(x(t_1))$,得到 $d_1=\frac{\tilde{I}_1(x(t_1))}{\Gamma(\gamma)}+c_1$,$d_2=\frac{1}{\Gamma(\gamma-1)}(\bar{I}_1(x(t_1))-\tilde{I}_1(x(t_1))t_1)+c_2$,$d_3=t_1^{3-\gamma}\times\left[I_1(x(t_1))-\frac{\tilde{I}_1(x(t_1))}{\Gamma(\gamma)}t_1^{\gamma-1}+\frac{\tilde{I}_1(x(t_1))}{\Gamma(\gamma-1)}t_1^{\gamma-1}-\frac{\bar{I}_1(x(t_1))}{\Gamma(\gamma-1)}t_1^{\gamma-2}\right]$。

因此 $t\in J_1$ 时,有

$$x(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s+c_1t^{\gamma-1}+c_2t^{\gamma-2}+\frac{\tilde{I}_1(x(t_1))}{\Gamma(\gamma)}(t-t_1)[t-(\gamma-2)t_1]t^{\gamma-3}+\frac{\bar{I}_1(x(t_1))}{\Gamma(\gamma-1)}(t-t_1)t^{\gamma-3}+I_1(x(t_1))t_1^{3-\gamma}t^{\gamma-3},$$

$$D_{0+}^{\gamma-2}x(t)=\int_0^t\frac{(t-s)^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}y(s)\mathrm{d}s+c_1\Gamma(\gamma)t+c_2\Gamma(\gamma-1)+\tilde{I}_1(x(t_1))(t-t_1)+\bar{I}_1(x(t_1)),$$

$$D_{0+}^{\gamma-1}x(t)=c_1\Gamma(\gamma)+\tilde{I}_1(x(t_1))+\int_0^t\frac{(t-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}y(s)\mathrm{d}s。$$

依次类推,当 $t\in J_k$ 时,可以得到

$$x(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s+c_1t^{\gamma-1}+c_2t^{\gamma-2}+t^{\gamma-3}\sum_{i=1}^k\frac{\tilde{I}_i(x(t_i))}{\Gamma(\gamma)}(t-t_i)[t-(\gamma-2)t_i]+t^{\gamma-3}\sum_{i=1}^k\frac{\bar{I}_i(x(t_i))}{\Gamma(\gamma-1)}(t-t_i)+t^{\gamma-3}\sum_{i=1}^kI_i(x(t_i))t_i^{3-\gamma},$$

$$D_{0+}^{\gamma-2}x(t)=\int_0^t\frac{(t-s)^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}y(s)\mathrm{d}s+c_1\Gamma(\gamma)t+c_2\Gamma(\gamma-1)+\sum_{i=1}^k\tilde{I}_i(x(t_i))(t-t_i)+\sum_{i=1}^k\bar{I}_i(x(t_i)),$$

$$D_{0+}^{\gamma-1}x(t)=\int_0^t\frac{(t-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}y(s)\mathrm{d}s+c_1\Gamma(\gamma)+\sum_{i=1}^k\tilde{I}_i(x(t_i))。$$

因此,对 $\forall t\in J$,有

$$x(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s+c_1t^{\gamma-1}+c_2t^{\gamma-2}+t^{\gamma-3}\sum_{0<t_k<t}\frac{\tilde{I}_k(x(t_k))}{\Gamma(\gamma)}(t-t_k)[t-(\gamma-2)t_k]+t^{\gamma-3}\sum_{0<t_k<t}\frac{\bar{I}_k(x(t_k))}{\Gamma(\gamma-1)}(t-t_k)+t^{\gamma-3}\sum_{0<t_k<t}I_k(x(t_k))t_k^{3-\gamma},\tag{5}$$

$$D_{0+}^{\gamma-2}x(t)=\int_0^t\frac{(t-s)^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}y(s)\mathrm{d}s+c_1\Gamma(\gamma)t+c_2\Gamma(\gamma-1)+\sum_{0<t_k<t}\tilde{I}_k(x(t_k))(t-t_k)+\sum_{0<t_k<t}\bar{I}_k(x(t_k)),$$

$$D_{0+}^{\gamma-1}x(t)=\int_0^t\frac{(t-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}y(s)\mathrm{d}s+c_1\Gamma(\gamma)+\sum_{0<t_k<t}\tilde{I}_k(x(t_k))。$$

利用边界条件 $D_{0+}^{\gamma-2}x(0)=aD_{0+}^{\gamma-1}x(1)$ 和 $D_{0+}^{\gamma-2}x(1)=bD_{0+}^{\gamma-1}x(0)$,可得

$$c_1=\frac{1}{(b-a-1)\Gamma(\gamma)}\left[\int_0^1\frac{(1-s)^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}y(s)\mathrm{d}s+a\int_0^1\frac{(1-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}y(s)\mathrm{d}s+a\sum_{k=1}^m\tilde{I}_k(x(t_k))+\sum_{k=1}^m\tilde{I}_k(x(t_k))(1-t_k)+\sum_{k=1}^m\bar{I}_k(x(t_k))\right],$$

$$c_2=\frac{a}{(b-a-1)\Gamma(\gamma-1)}\left[\int_0^1\frac{(1-s)^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}y(s)\mathrm{d}s+(b-1)\int_0^1\frac{(1-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}y(s)\mathrm{d}s+(b-1)\sum_{k=1}^m\tilde{I}_k(x(t_k))+\sum_{k=1}^m\tilde{I}_k(x(t_k))(1-t_k)+\sum_{k=1}^m\bar{I}_k(x(t_k))\right]。$$

将 c_1 和 c_2 代入式(5),可得式(3)。反之,容易验证得出式(3)满足给定边界条件下的边值问题(2),故引理得证。

2 主要结果

定义算子 $T:PC_\gamma^1[J,R]\rightarrow PC_\gamma^1[J,R]$ 如下:

$$\begin{aligned}(Tx)(t)=&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s),D_{0+}^{\gamma-2}x(s),D_{0+}^{\gamma-1}x(s))\mathrm{d}s+\\&\frac{t^{\gamma-1}+a(\gamma-1)t^{\gamma-2}}{(b-a-1)\Gamma(\gamma)}\int_0^1\frac{(1-s)^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}f(s,x(s),D_{0+}^{\gamma-2}x(s),D_{0+}^{\gamma-1}x(s))\mathrm{d}s+\\&\frac{a[t^{\gamma-1}+(b-1)(\gamma-1)t^{\gamma-2}]}{(b-a-1)\Gamma(\gamma)}\int_0^1\frac{(1-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}f(s,x(s),D_{0+}^{\gamma-2}x(s),D_{0+}^{\gamma-1}x(s))\mathrm{d}s+\\&\frac{a[t^{\gamma-1}+(b-1)(\gamma-1)t^{\gamma-2}]}{(b-a-1)\Gamma(\gamma)}\sum_{k=1}^m\tilde{I}_k(x(t_k))+\frac{t^{\gamma-1}+a(\gamma-1)t^{\gamma-2}}{(b-a-1)\Gamma(\gamma)}\times\\&\sum_{k=1}^m[\tilde{I}_k(x(t_k))(1-t_k)+\bar{I}_k(x(t_k))]+t^{\gamma-3}\sum_{0<t_k<t}\left\{\frac{\tilde{I}_k(x(t_k))}{\Gamma(\gamma)}(t-t_k)[t-(\gamma-2)t_k]+\right.\\&\left.\frac{\bar{I}_k(x(t_k))}{\Gamma(\gamma-1)}(t-t_k)+I_k(x(t_k))t_k{}^{3-\gamma}\right\},\end{aligned}$$

$$\begin{aligned}(T_1x)(t)=&\frac{t^{\gamma-1}+a(\gamma-1)t^{\gamma-2}}{(b-a-1)\Gamma(\gamma)}\int_0^1\frac{(1-s)^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}f(s,x(s),D_{0+}^{\gamma-2}x(s),D_{0+}^{\gamma-1}x(s))\mathrm{d}s+\\&\frac{a[t^{\gamma-1}+(b-1)(\gamma-1)t^{\gamma-2}]}{(b-a-1)\Gamma(\gamma)}\int_0^1\frac{(1-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}f(s,x(s),D_{0+}^{\gamma-2}x(s),D_{0+}^{\gamma-1}x(s))\mathrm{d}s+\\&\frac{a[t^{\gamma-1}+(b-1)(\gamma-1)t^{\gamma-2}]}{(b-a-1)\Gamma(\gamma)}\sum_{k=1}^m\tilde{I}_k(x(t_k))+\frac{t^{\gamma-1}+a(\gamma-1)t^{\gamma-2}}{(b-a-1)\Gamma(\gamma)}\times\\&\sum_{k=1}^m[\tilde{I}_k(x(t_k))(1-t_k)+\bar{I}_k(x(t_k))]+t^{\gamma-3}\sum_{0<t_k<t}\left\{\frac{\tilde{I}_k(x(t_k))}{\Gamma(\gamma)}(t-t_k)[t-(\gamma-2)t_k]+\right.\\&\left.\frac{\bar{I}_k(x(t_k))}{\Gamma(\gamma-1)}(t-t_k)+I_k(x(t_k))t_k{}^{3-\gamma}\right\},\end{aligned}$$

$$(T_2x)(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s),D_{0+}^{\gamma-2}x(s),D_{0+}^{\gamma-1}x(s))\mathrm{d}s,$$

则 $T=T_1+T_2$。显然,x 是边值问题(1)的解当且仅当 x 是算子 $T=T_1+T_2$ 的不动点。根据引理4,得到

$$\begin{aligned}D_{0+}^{\gamma-2}(T_1x)(t)=&\frac{a+t}{b-a-1}\int_0^1\frac{(1-s)^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}f(s,x(s),D_{0+}^{\gamma-2}x(s),D_{0+}^{\gamma-1}x(s))\mathrm{d}s+\\&\frac{a(b-1+t)}{b-a-1}\int_0^1\frac{(1-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}f(s,x(s),D_{0+}^{\gamma-2}x(s),D_{0+}^{\gamma-1}x(s))\mathrm{d}s+\\&\frac{a(b-1+t)}{b-a-1}\sum_{k=1}^m\tilde{I}_k(x(t_k))+\frac{a+t}{b-a-1}\sum_{k=1}^m[\tilde{I}_k(x(t_k))(1-t_k)+\bar{I}_k(x(t_k))]+\\&\sum_{0<t_k<t}[\tilde{I}_k(x(t_k))(t-t_k)+\bar{I}_k(x(t_k))],\end{aligned}$$

$$D_{0+}^{\gamma-2}(T_2x)(t)=\int_0^t\frac{(t-s)^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}f(s,x(s),D_{0+}^{\gamma-2}x(s),D_{0+}^{\gamma-1}x(s))\mathrm{d}s,$$

$$\begin{aligned}D_{0+}^{\gamma-1}(T_1x)(t)=&\frac{1}{b-a-1}\int_0^1\frac{(1-s)^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}f(s,x(s),D_{0+}^{\gamma-2}x(s),D_{0+}^{\gamma-1}x(s))\mathrm{d}s+\\&\frac{a}{b-a-1}\int_0^1\frac{(1-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}f(s,x(s),D_{0+}^{\gamma-2}x(s),D_{0+}^{\gamma-1}x(s))\mathrm{d}s+\frac{a}{b-a-1}\sum_{k=1}^m\tilde{I}_k(x(t_k))+\\&\frac{1}{b-a-1}\sum_{k=1}^m[\tilde{I}_k(x(t_k))(1-t_k)+\bar{I}_k(x(t_k))]+\sum_{0<t_k<t}\tilde{I}_k(x(t_k)),\end{aligned}$$

$$D_{0+}^{\gamma-1}(T_2x)(t)=\int_0^t\frac{(t-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}f(s,x(s),D_{0+}^{\gamma-2}x(s),D_{0+}^{\gamma-1}x(s))\mathrm{d}s。$$

假设 $f:J\times R^3\to R$ 连续，$I_k,\bar{I}_k,\tilde{I}_k\in C[R,R]$，$k=1,2,\cdots,m$，$f(t,x,y,z)$ 和 $I_k,\bar{I}_k,\tilde{I}_k$ 满足以下条件：

(H_1) 存在常数 $R>0$ 和单调不减函数 $h(t)\geqslant 0$，且 $(1-t)^{\alpha-\gamma}h(t)\in L[0,1]$，使得对 $\forall t\in J,x,y,z\in\Omega_R$，有 $|f(t,x,y,z)|\leqslant h(t)$，且 $H_R=\int_0^1(1-t)^{\alpha-\gamma}h(t)\mathrm{d}t\leqslant\frac{1}{3}R$，其中 $\Omega_R=\{(x,y,z)\mid\max\{|x|,|y|,|z|\}\leqslant R\}$；

(H') 存在非负且单调不减的函数 $g(t)$，并且 $(1-t)^{\alpha-\gamma}g(t)\in L[0,1]$，使得对 $\forall t\in J,x,y,z\in R$，有 $|f(t,x,y,z)|\leqslant g(t)$，且 $G=\int_0^1(1-t)^{\alpha-\gamma}g(t)\mathrm{d}t$；

(H_2) 存在常数 $c_k,d_k,e_k>0$，使得对 $\forall t\in J,x\in R$，有

$|I_k(x)|\leqslant c_k|x|$，$|\bar{I}_k(x)|\leqslant d_k|x|$，$|\tilde{I}_k(x)|\leqslant e_k|x|$，且 $c^*=\sum_{k=1}^m c_k$，$d^*=\sum_{k=1}^m d_k$，$e^*=\sum_{k=1}^m e_k$；

(H_3) 存在常数 $K_i>0$，$i=1,2,3$，使得对 $\forall t\in J,x,\bar{x},y,\bar{y}z,\bar{z}\in R$，有

$|f(t,x,y,z)-f(t,\bar{x},\bar{y},\bar{z})|\leqslant K_1|x-\bar{x}|+K_2|y-\bar{y}|+K_3|z-\bar{z}|$。

定理 1　假设条件(H_1)－(H_3)成立，且满足条件 $m=\dfrac{\sum_{i=1}^3 K_i}{\min\{\Gamma(\alpha+1),\Gamma(\alpha-\gamma+2),\Gamma(\alpha-\gamma+3)\}}<1$ 以及 $M=\max\{M_1,M_2\}\leqslant 1$，其中

$$M_1=\frac{1}{3\Gamma(\alpha)}+\frac{1+a(\gamma-1)+a[1+(b-1)(\gamma-1)](\alpha-\gamma+1)}{3(b-a-1)\Gamma(\gamma)\Gamma(\alpha-\gamma+2)}+\frac{(b-a-1)\Gamma(\gamma)c^*+[1+(b-1)(\gamma-1)]d^*+b[1+a(\gamma-1)]e^*}{(b-a-1)\Gamma(\gamma)},$$

$$M_2=\frac{b[1+a(\alpha-\gamma+1)]+3b\Gamma(\alpha-\gamma+2)[d^*+(a+1)e^*]}{3(b-a-1)\Gamma(\alpha-\gamma+2)},$$

那么边值问题(1)至少存在 1 个解。

证明：对 $\forall x,y\in\Omega_R$，$\forall t\in J$，由条件(H_1)和(H_2)，得到

$$\begin{aligned}|(T_1x)(t)+(T_2y)(t)|\leqslant&\frac{[1+a(\gamma-1)]\|h\|_L}{(b-a-1)\Gamma(\gamma)\Gamma(\alpha-\gamma+2)}+\frac{a[1+(b-1)(\gamma-1)]H_R}{(b-a-1)\Gamma(\gamma)\Gamma(\alpha-\gamma+1)}+\\&\frac{a[1+(b-1)(\gamma-1)]}{(b-a-1)\Gamma(\gamma)}e^*\|x\|+\frac{1+a(\gamma-1)}{(b-a-1)\Gamma(\gamma)}(d^*+e^*)\|x\|+\\&\left(\frac{e^*}{\Gamma(\gamma)}+\frac{d^*}{\Gamma(\gamma-1)}+c^*\right)\|x\|+\frac{\|h\|_L}{\Gamma(\alpha)}\leqslant\\&\left\{\frac{1}{3\Gamma(\alpha)}+\frac{1+a(\gamma-1)+a[1+(b-1)(\gamma-1)](\alpha-\gamma+1)}{3(b-a-1)\Gamma(\gamma)\Gamma(\alpha-\gamma+2)}+\right.\\&\left.\frac{(b-a-1)\Gamma(\gamma)c^*+[1+(b-1)(\gamma-1)]d^*+b[1+a(\gamma-1)]e^*}{(b-a-1)\Gamma(\gamma)}\right\}R=\\&M_1R\leqslant MR\leqslant R,\end{aligned}$$

故 $\|T_1x+T_2y\|_\infty\leqslant R$。

$$\begin{aligned}|D_{0+}^{\gamma-2}(T_1x)(t)+D_{0+}^{\gamma-2}(T_2y)(t)|\leqslant&\frac{(a+1)\|h\|_L}{(b-a-1)\Gamma(\alpha-\gamma+2)}+\frac{abH_R}{(b-a-1)\Gamma(\alpha-\gamma+1)}+\\&\frac{ab}{b-a-1}e^*\|x\|+\frac{a+1}{b-a-1}(d^*+e^*)\|x\|+(d^*+e^*)\|x\|+\frac{\|h\|_L}{\Gamma(\alpha-\gamma+2)}\leqslant\\&\frac{b[1+a(\alpha-\gamma+1)]+3b\Gamma(\alpha-\gamma+2)[d^*+(a+1)e^*]}{3(b-a-1)\Gamma(\alpha-\gamma+2)}R=M_2R\leqslant MR\leqslant R,\end{aligned}$$

所以 $\|D_{0+}^{\gamma-2}(T_1x)+D_{0+}^{\gamma-2}(T_2y)\|_\infty\leqslant R$。

$$\begin{aligned}|D_{0+}^{\gamma-1}(T_1x)(t)+D_{0+}^{\gamma-1}(T_2y)(t)|\leqslant&\frac{1}{b-a-1}\int_0^1\frac{1}{\Gamma(\alpha-\gamma+2)}h(s)\mathrm{d}s+\frac{a}{b-a-1}\times\\&\int_0^1\frac{(1-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}h(s)\mathrm{d}s+\frac{a}{b-a-1}\sum_{k=1}^m e_k|x(t_k)|+\frac{1}{b-a-1}\sum_{k=1}^m[e_k|x(t_k)|+d_k|x(t_k)|]+\\&\sum_{k=1}^m e_k|x(t_k)|+\int_0^t\frac{(t-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}h(s)\mathrm{d}s,\end{aligned}$$

对$\int_0^t \frac{(t-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}h(s)\mathrm{d}s$进行变量代换$s=tu$，并利用函数$h(t)$的单调不减性，可得

$$\int_0^t \frac{(t-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}h(s)\mathrm{d}s = t^{\alpha-\gamma+1}\int_0^1 \frac{(1-u)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}h(tu)\mathrm{d}u \leqslant$$
$$t^{\alpha-\gamma+1}\int_0^1 \frac{(1-u)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}h(u)\mathrm{d}u \leqslant \frac{H_R}{\Gamma(\alpha-\gamma+1)},$$

故$|D_{0+}^{\gamma-1}(T_1x)(t)+D_{0+}^{\gamma-1}(T_2y)(t)| \leqslant \frac{\|h\|_L}{(b-a-1)\Gamma(\alpha-\gamma+2)}+\frac{aH_R}{(b-a-1)\Gamma(\alpha-\gamma+1)}+$
$\frac{a}{b-a-1}e^*\|x\|+\frac{1}{b-a-1}(d^*+e^*)\|x\|+e^*\|x\|+\frac{H_R}{\Gamma(\alpha-\gamma+1)} \leqslant$

$$\frac{b[1+a(\alpha-\gamma+1)]+3b\Gamma(\alpha-\gamma+2)[d^*+(a+1)e^*]}{3(b-a-1)\Gamma(\alpha-\gamma+2)}R = M_2R \leqslant MR \leqslant R,$$

$\|D_{0+}^{\gamma-1}(T_1x)+D_{0+}^{\gamma-1}(T_2y)\|_\infty \leqslant R$。所以$\|T_1x+T_2y\| \leqslant R$，$T_1x+T_2y \in \Omega_R$。

证明算子T_1的全连续性。因为$f, I_k, \bar{I}_k, \tilde{I}_k$是连续函数，所以算子$T_1$是连续的。证明算子$T_1$在$\Omega_R$上是一致有界的。对$\forall t \in J, \forall x \in \Omega_R$，根据条件($\mathrm{H}_1$)和($\mathrm{H}_2$)，可得

$$|(T_1x)(t)| \leqslant \frac{[1+a(\gamma-1)]\|h\|_L}{(b-a-1)\Gamma(\gamma)\Gamma(\alpha-\gamma+2)}+\frac{a[1+(b-1)(\gamma-1)]H_R}{(b-a-1)\Gamma(\gamma)\Gamma(\alpha-\gamma+1)}+$$
$$\frac{a[1+(b-1)(\gamma-1)]}{(b-a-1)\Gamma(\gamma)}e^*\|x\|+\frac{1+a(\gamma-1)}{(b-a-1)\Gamma(\gamma)}(d^*+e^*)\|x\|+$$
$$\left(\frac{e^*}{\Gamma(\gamma)}+\frac{d^*}{\Gamma(\gamma-1)}+c^*\right)\|x\| \leqslant \left\{\frac{1+a(\gamma-1)+a[1+(b-1)(\gamma-1)](\alpha-\gamma+1)}{3(b-a-1)\Gamma(\gamma)\Gamma(\alpha-\gamma+2)}+\right.$$
$$\left.\frac{(b-a-1)\Gamma(\gamma)c^*+[1+(b-1)(\gamma-1)]d^*+b[1+a(\gamma-1)]e^*}{(b-a-1)\Gamma(\gamma)}\right\}R := N_1,$$

故$\|T_1x\|_\infty \leqslant N_1$。

$$|D_{0+}^{\gamma-2}(T_1x)(t)| \leqslant \frac{(a+1)\|h\|_L}{(b-a-1)\Gamma(\alpha-\gamma+2)}+\frac{abH_R}{(b-a-1)\Gamma(\alpha-\gamma+1)}+\frac{ab}{b-a-1}e^*\|x\|+$$
$$\frac{a+1}{b-a-1}(d^*+e^*)\|x\|+(d^*+e^*)\|x\| \leqslant$$
$$\frac{1+a[1+b(\alpha-\gamma+1)]+3b\Gamma(\alpha-\gamma+2)[d^*+(a+1)e^*]}{3(b-a-1)\Gamma(\alpha-\gamma+2)}R := N_2,$$

所以$\|D_{0+}^{\gamma-2}(T_1x)\|_\infty \leqslant N_2$。

$$|D_{0+}^{\gamma-1}(T_1x)(t)| \leqslant \frac{\|h\|_L}{(b-a-1)\Gamma(\alpha-\gamma+2)}+\frac{aH_R}{(b-a-1)\Gamma(\alpha-\gamma+1)}+\frac{a}{b-a-1}e^*\|x\|+$$
$$\frac{1}{b-a-1}(d^*+e^*)\|x\|+e^*\|x\| \leqslant$$
$$\frac{1+a[1+b(\alpha-\gamma+1)]+3b\Gamma(\alpha-\gamma+2)[d^*+(a+1)e^*]}{3(b-a-1)\Gamma(\alpha-\gamma+2)}R := N_2,$$

故也可以得出$\|D_{0+}^{\gamma-1}(T_1x)\|_\infty \leqslant N_2$。

取$N=\max\{N_1,N_2\}$，所以$\|T_1x\| \leqslant N$，因此T_1在Ω_R上一致有界。

证明算子T_1在Ω_R上是等度连续的。如果$\tau_1,\tau_2 \in J_0$，且$\tau_1 < \tau_2$，则

$$|(T_1x)(\tau_1)-(T_1x)(\tau_2)| \leqslant \frac{|\tau_1^{\gamma-1}-\tau_2^{\gamma-1}|}{(b-a-1)\Gamma(\gamma)} \times$$
$$\left\{\frac{\|h\|_L}{\Gamma(\alpha-\gamma+2)}+\frac{aH_R}{\Gamma(\alpha-\gamma+1)}+[d^*+(a+1)e^*]R\right\}+$$
$$\frac{a|\tau_1^{\gamma-2}-\tau_2^{\gamma-2}|}{(b-a-1)\Gamma(\gamma-1)}\left\{\frac{\|h\|_L}{\Gamma(\alpha-\gamma+2)}+\frac{(b-1)H_R}{\Gamma(\alpha-\gamma+1)}+(d^*+be^*)R\right\},$$

因为$\tau^{\gamma-i}(i=1,2)$在$\tau \in J_0$上连续，从而一致连续。所以，当$\tau_1 \to \tau_2$时，$|(T_1x)(\tau_1)-(T_1x)(\tau_2)| \to 0$。

$$|D_{0+}^{\gamma-2}(T_1x)(\tau_1)-D_{0+}^{\gamma-2}(T_1x)(\tau_2)| \leqslant \frac{|\tau_1-\tau_2|}{b-a-1}\left\{\frac{\|h\|_L}{\Gamma(\alpha-\gamma+2)}+\frac{aH_R}{\Gamma(\alpha-\gamma+1)}+[d^*+(a+1)e^*]R\right\},$$

利用 τ 在区间 J_0 上的一致连续性，可得当 $\tau_1 \to \tau_2$ 时，

$$|D_{0+}^{\gamma-2}(T_1x)(\tau_1) - D_{0+}^{\gamma-2}(T_1x)(\tau_2)| \to 0。$$

因为 $|D_{0+}^{\gamma-1}(T_1x)(\tau_1) - D_{0+}^{\gamma-1}(T_1x)(\tau_2)| = 0$，所以，当 $\tau_1 \to \tau_2$ 时，

$$|D_{0+}^{\gamma-1}(T_1x)(\tau_1) - D_{0+}^{\gamma-1}(T_1x)(\tau_2)| \to 0。$$

如果 $\tau_1, \tau_2 \in J_k, k=1,2,\cdots,m$，且 $\tau_1 < \tau_2$，则

$$|(T_1x)(\tau_1) - (T_1x)(\tau_2)| \leqslant \frac{|\tau_1{}^{\gamma-1} - \tau_2{}^{\gamma-1}|}{(b-a-1)\Gamma(\gamma)}\left\{\frac{\|h\|_L}{\Gamma(\alpha-\gamma+2)} + \frac{aH_R}{\Gamma(\alpha-\gamma+1)} + (d^* + be^*)R\right\} +$$

$$\frac{|\tau_1{}^{\gamma-2} - \tau_2{}^{\gamma-2}|}{(b-a-1)\Gamma(\gamma-1)}\left\{\frac{a\|h\|_L}{\Gamma(\alpha-\gamma+2)} + \frac{a(b-1)H_R}{\Gamma(\alpha-\gamma+1)} + [(b-1)d^* + (a+1)(b-1)e^*]R\right\} +$$

$$|\tau_1{}^{\gamma-3} - \tau_2{}^{\gamma-3}|\left[\frac{(\gamma-2)e^*}{\Gamma(\gamma)} + \frac{d^*}{\Gamma(\gamma-1)} + c^*\right]R,$$

由于 $\tau^{\gamma-i}(i=1,2,3)$ 在 $\tau \in \bar{J}_k = [t_k, t_{k+1}](k=1,2,\cdots,m)$ 上连续，从而一致连续，所以在 $\tau \in J_k$ 上也是一致连续的，故当 $\tau_1 \to \tau_2$ 时，$|(T_1x)(\tau_1) - (T_1x)(\tau_2)| \to 0$。

$$|D_{0+}^{\gamma-2}(T_1x)(\tau_1) - D_{0+}^{\gamma-2}(T_1x)(\tau_2)| \leqslant$$

$$\frac{|\tau_1 - \tau_2|}{b-a-1}\left\{\frac{\|h\|_L}{\Gamma(\alpha-\gamma+2)} + \frac{aH_R}{\Gamma(\alpha-\gamma+1)} + (d^* + be^*)R\right\},$$

因为 τ 在 $\bar{J}_k$ 上一致连续，故在 J_k 上一致连续，所以，当 $\tau_1 \to \tau_2$ 时，$|D_{0+}^{\gamma-2}(T_1x)(\tau_1) - D_{0+}^{\gamma-2}(T_1x)(\tau_2)| \to 0$。

由于 $|D_{0+}^{\gamma-1}(T_1x)(\tau_1) - D_{0+}^{\gamma-1}(T_1x)(\tau_2)| = 0$，所以，当 $\tau_1 \to \tau_2$ 时，$|D_{0+}^{\gamma-1}(T_1x)(\tau_1) - D_{0+}^{\gamma-1}(T_1x)(\tau_2)| \to 0$，故算子 T_1 在 Ω_R 上等度连续。根据 Arzela-Ascoli 定理可得 T_1 在 Ω_R 上是紧的，因此，T_1 是全连续算子。

以下证明 T_2 为压缩映射。对 $\forall x, y \in \Omega_R, \forall t \in [0,1]$，由条件$(H_3)$，得到

$$|(T_2x)(t) - (T_2y)(t)| \leqslant$$

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s),D_{0+}^{\gamma-2}x(s),D_{0+}^{\gamma-1}x(s)) - f(s,y(s),D_{0+}^{\gamma-2}y(s),D_{0+}^{\gamma-1}y(s))|\mathrm{d}s \leqslant$$

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(K_1|x(s)-y(s)| + K_2|D_{0+}^{\gamma-2}x(s) - D_{0+}^{\gamma-2}y(s)| + K_3|D_{0+}^{\gamma-1}x(s) - D_{0+}^{\gamma-1}y(s)|)\mathrm{d}s \leqslant$$

$$\frac{\sum_{i=1}^{3}K_i}{\Gamma(\alpha+1)}\|x-y\|,$$

$$|D_{0+}^{\gamma-2}(T_2x)(t) - D_{0+}^{\gamma-2}(T_2y)(t)| \leqslant \int_0^t \frac{(t-s)^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}(K_1|x(s)-y(s)| +$$

$$K_2|D_{0+}^{\gamma-2}x(s) - D_{0+}^{\gamma-2}y(s)| + K_3|D_{0+}^{\gamma-1}x(s) - D_{0+}^{\gamma-1}y(s)|)\mathrm{d}s \leqslant \frac{\sum_{i=1}^{3}K_i}{\Gamma(\alpha-\gamma+3)}\|x-y\|,$$

$$|D_{0+}^{\gamma-1}(T_2x)(t) - D_{0+}^{\gamma-1}(T_2y)(t)| \leqslant \int_0^t \frac{(t-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}(K_1|x(s)-y(s)| + K_2|D_{0+}^{\gamma-2}x(s) - D_{0+}^{\gamma-2}y(s)| +$$

$$K_3|D_{0+}^{\gamma-1}x(s) - D_{0+}^{\gamma-1}y(s)|)\mathrm{d}s \leqslant \frac{\sum_{i=1}^{3}K_i}{\Gamma(\alpha-\gamma+2)}\|x-y\|。$$

因此，$\|T_2x - T_2y\| \leqslant \dfrac{\sum_{i=1}^{3}K_i}{\min\{\Gamma(\alpha+1),\Gamma(\alpha-\gamma+2),\Gamma(\alpha-\gamma+3)\}}\|x-y\| = m\|x-y\|$。又因为 $m < 1$，所以 T_2 是压缩算子。根据引理 3 可知，边值问题(1) 在 J 上至少有 1 个解。

定理 2　假设条件$(H'_1)-(H_3)$成立，且满足条件 $m = \dfrac{\sum_{i=1}^{3}K_i}{\min\{\Gamma(\alpha+1),\Gamma(\alpha-\gamma+2),\Gamma(\alpha-\gamma+3)\}} < 1$ 以及 $\rho = \max\{\rho_1, \rho_2\} < 1$，其中 $\rho_1 = c^* + \dfrac{(1+(b-1)(\gamma-1))d^* + b(1+a(\gamma-1))e^*}{(b-a-1)\Gamma(\gamma)}$，$\rho_2 =$

$\dfrac{bd^*+b(a+1)e^*}{b-a-1}$,则边值问题(1)至少存在 1 个解。

证明:取 $B_r=\{x\in PC_\gamma^1[J,\mathbf{R}]:\|x\|\leqslant r\}$,其中

$$r\geqslant\max\left\{\lambda_1\Big/\left\{1-\left[c^*+\frac{(1+(b-1)(\gamma-1))d^*+b(1+a(\gamma-1))e^*}{(b-a-1)\Gamma(\gamma)}\right]\right\},\lambda_2\Big/\left[1-\frac{bd^*+b(a+1)e^*}{b-a-1}\right]\right\},$$

$$\lambda_1=\left[\frac{1}{\Gamma(\alpha)}+\frac{1+a(\gamma-1)}{(b-a-1)\Gamma(\gamma)\Gamma(\alpha-\gamma+2)}\right]\|g\|_L+\frac{a[1+(b-1)(\gamma-1)]G}{(b-a-1)\Gamma(\gamma)\Gamma(\alpha-\gamma+1)},$$

$$\lambda_2=\frac{b\|g\|_L}{(b-a-1)\Gamma(\alpha-\gamma+2)}+\frac{abG}{(b-a-1)\Gamma(\alpha-\gamma+1)}。$$

首先,对 $\forall x,y\in B_r,\forall t\in J$,根据条件($\mathrm{H}'_1$)和($\mathrm{H}_2$),可得

$$|(T_1x)(t)+(T_2y)(t)|\leqslant\lambda_1+\left[c^*+\frac{(1+(b-1)(\gamma-1))d^*+b(1+a(\gamma-1))e^*}{(b-a-1)\Gamma(\gamma)}\right]\|x\|\leqslant$$
$$\left\{1-\left[c^*+\frac{(1+(b-1)(\gamma-1))d^*+b(1+a(\gamma-1))e^*}{(b-a-1)\Gamma(\gamma)}\right]\right\}r+$$
$$\left[c^*+\frac{(1+(b-1)(\gamma-1))d^*+b(1+a(\gamma-1))e^*}{(b-a-1)\Gamma(\gamma)}\right]r=r,$$故 $\|T_1x+T_2y\|_\infty\leqslant r$。

$$|D_{0+}^{\gamma-2}(T_1x)(t)+D_{0+}^{\gamma-2}(T_2y)(t)|\leqslant\lambda_2+\frac{bd^*+b(a+1)e^*}{b-a-1}\|x\|\leqslant\left[1-\frac{bd^*+b(a+1)e^*}{b-a-1}\right]r+$$
$$\frac{bd^*+b(a+1)e^*}{b-a-1}r=r,$$故 $\|D_{0+}^{\gamma-2}(T_1x)+D_{0+}^{\gamma-2}(T_2y)\|_\infty\leqslant r$。

$$|D_{0+}^{\gamma-1}(T_1x)(t)+D_{0+}^{\gamma-1}(T_2y)(t)|\leqslant\frac{1}{b-a-1}\int_0^1\frac{1}{\Gamma(\alpha-\gamma+2)}g(s)\mathrm{d}s+$$
$$\frac{a}{b-a-1}\int_0^1\frac{(1-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}g(s)\mathrm{d}s+\frac{a}{b-a-1}\sum_{k=1}^m e_k|x(t_k)|+$$
$$\frac{1}{b-a-1}\sum_{k=1}^m[e_k|x(t_k)|+d_k|x(t_k)|]+\sum_{k=1}^m e_k|x(t_k)|+\int_0^t\frac{(t-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}g(s)\mathrm{d}s。$$

类似定理 1,令 $s=tu$,可得

$$\int_0^t\frac{(t-s)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}g(s)\mathrm{d}s=t^{\alpha-\gamma+1}\int_0^1\frac{(1-u)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}g(tu)\mathrm{d}u\leqslant t^{\alpha-\gamma+1}\int_0^1\frac{(1-u)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}g(u)\mathrm{d}u\leqslant\frac{G}{\Gamma(\alpha-\gamma+1)},$$

则 $|D_{0+}^{\gamma-1}(T_1x)(t)+D_{0+}^{\gamma-1}(T_2y)(t)|\leqslant$

$$\frac{\|g\|_L}{(b-a-1)\Gamma(\alpha-\gamma+2)}+\frac{(b-1)G}{(b-a-1)\Gamma(\alpha-\gamma+1)}+\frac{d^*+be^*}{b-a-1}\|x\|\leqslant$$
$$\lambda_2+\frac{bd^*+b(a+1)e^*}{b-a-1}\|x\|\leqslant\left[1-\frac{bd^*+b(a+1)e^*}{b-a-1}\right]r+\frac{bd^*+b(a+1)e^*}{b-a-1}r=r,$$

故 $\|D_{0+}^{\gamma-1}(T_1x)+D_{0+}^{\gamma-1}(T_2y)\|_\infty\leqslant r$,$\|T_1x+T_2y\|\leqslant r$,因此 $T_1x+T_2y\in B_r$。

其次,证明 T_1 的全连续性。显然,算子 T_1 连续。对 $\forall t\in J,\forall x\in B_r$,由条件($\mathrm{H}'_1$)和($\mathrm{H}_2$),得到

$$|(T_1x)(t)|\leqslant\frac{[1+a(\gamma-1)]\|g\|_L}{(b-a-1)\Gamma(\gamma)\Gamma(\alpha-\gamma+2)}+\frac{a[1+(b-1)(\gamma-1)]G}{(b-a-1)\Gamma(\gamma)\Gamma(\alpha-\gamma+1)}+$$
$$\left\{c^*+\frac{[1+(b-1)(\gamma-1)]d^*+b[1+a(\gamma-1)]e^*}{(b-a-1)\Gamma(\gamma)}\right\}r:=L_1,$$因此 $\|T_1x\|_\infty\leqslant L_1$。

$|D_{0+}^{\gamma-2}(T_1x)(t)|\leqslant\dfrac{(a+1)\|g\|_L}{(b-a-1)\Gamma(\alpha-\gamma+2)}+\dfrac{abG}{(b-a-1)\Gamma(\alpha-\gamma+1)}+\dfrac{bd^*+b(a+1)e^*}{b-a-1}r:=L_2$,因此 $\|D_{0+}^{\gamma-2}(T_1x)\|_\infty\leqslant L_2$。$|D_{0+}^{\gamma-1}(T_1x)(t)|\leqslant\dfrac{\|g\|_L}{(b-a-1)\Gamma(\alpha-\gamma+2)}+\dfrac{aG}{(b-a-1)\Gamma(\alpha-\gamma+1)}+$
$\dfrac{d^*+be^*}{b-a-1}r\leqslant\dfrac{(a+1)\|g\|_L}{(b-a-1)\Gamma(\alpha-\gamma+2)}+\dfrac{abG}{(b-a-1)\Gamma(\alpha-\gamma+1)}+\dfrac{bd^*+b(a+1)e^*}{b-a-1}r:=L_2$,因此 $\|D_{0+}^{\gamma-1}(T_1x)\|_\infty\leqslant L_2$。令 $L=\max\{L_1,L_2\}$,$\|T_1x\|\leqslant L$,所以 T_1 在 B_r 上一致有界。若 $\tau_1,\tau_2\in J_0$,且 $\tau_1<\tau_2$,则

$$|(T_1x)(\tau_1)-(T_1x)(\tau_2)|\leqslant\frac{|\tau_1^{\gamma-1}-\tau_2^{\gamma-1}|}{(b-a-1)\Gamma(\gamma)}\left\{\frac{\|g\|_L}{\Gamma(\alpha-\gamma+2)}+\frac{aG}{\Gamma(\alpha-\gamma+1)}+[d^*+(a+1)e^*]r\right\}+\frac{a|\tau_1^{\gamma-2}-\tau_2^{\gamma-2}|}{(b-a-1)\Gamma(\gamma-1)}\left\{\frac{\|g\|_L}{\Gamma(\alpha-\gamma+2)}+\frac{(b-1)G}{\Gamma(\alpha-\gamma+1)}+(d^*+be^*)r\right\},$$

因为 $\tau^{\gamma-i}(i=1,2)$ 在 J_0 上一致连续,故当 $\tau_1\to\tau_2$ 时,$|(T_1x)(\tau_1)-(T_1x)(\tau_2)|\to0$。

$$|D_{0+}^{\gamma-2}(T_1x)(\tau_1)-D_{0+}^{\gamma-2}(T_1x)(\tau_2)|\leqslant\frac{|\tau_1-\tau_2|}{b-a-1}\left\{\frac{\|g\|_L}{\Gamma(\alpha-\gamma+2)}+\frac{aG}{\Gamma(\alpha-\gamma+1)}+[d^*+(a+1)e^*]r\right\},$$

由于 τ 在 J_0 上具有一致连续性,故当 $\tau_1\to\tau_2$ 时,$|D_{0+}^{\gamma-2}(T_1x)(\tau_1)-D_{0+}^{\gamma-2}(T_1x)(\tau_2)|\to0$。

因为 $|D_{0+}^{\gamma-1}(T_1x)(\tau_1)-D_{0+}^{\gamma-1}(T_1x)(\tau_2)|=0$,所以,当 $\tau_1\to\tau_2$ 时,$|D_{0+}^{\gamma-1}(T_1x)(\tau_1)-D_{0+}^{\gamma-1}(T_1x)(\tau_2)|\to0$。

若 $\tau_1,\tau_2\in J_k,k=1,2,\cdots,m$,且 $\tau_1<\tau_2$,则

$$\begin{aligned}|(T_1x)(\tau_1)-(T_1x)(\tau_2)|\leqslant&\frac{|\tau_1^{\gamma-1}-\tau_2^{\gamma-1}|}{(b-a-1)\Gamma(\gamma)}\left\{\frac{\|g\|_L}{\Gamma(\alpha-\gamma+2)}+\frac{aG}{\Gamma(\alpha-\gamma+1)}+(d^*+be^*)r\right\}+\\&\frac{|\tau_1^{\gamma-2}-\tau_2^{\gamma-2}|}{(b-a-1)\Gamma(\gamma-1)}\left\{\frac{a\|g\|_L}{\Gamma(\alpha-\gamma+2)}+\frac{a(b-1)G}{\Gamma(\alpha-\gamma+1)}+\right.\\&\left.[(b-1)d^*+(a+1)(b-1)e^*]r\right\}+\\&|\tau_1^{\gamma-3}-\tau_2^{\gamma-3}|\left[\frac{(\gamma-2)e^*}{\Gamma(\gamma)}+\frac{d^*}{\Gamma(\gamma-1)}+c^*\right]r。\end{aligned}$$

因为 $\tau^{\gamma-i}(i=1,2,3)$ 在 $\bar{J}_k=[t_k,t_{k+1}](k=1,2,\cdots,m)$ 上一致连续,所以在 J_k 上也一致连续,故当 $\tau_1\to\tau_2$ 时 $|(T_1x)(\tau_1)-(T_1x)(\tau_2)|\to0$。

$$|D_{0+}^{\gamma-2}(T_1x)(\tau_1)-D_{0+}^{\gamma-2}(T_1x)(\tau_2)|\leqslant\frac{|\tau_1-\tau_2|}{b-a-1}\left\{\frac{\|g\|_L}{\Gamma(\alpha-\gamma+2)}+\frac{aG}{\Gamma(\alpha-\gamma+1)}+(d^*+be^*)r\right\},$$

因为 τ 在 $\bar{J}_k$ 上一致连续,故在 J_k 上一致连续,所以,当 $\tau_1\to\tau_2$ 时,$|D_{0+}^{\gamma-2}(T_1x)(\tau_1)-D_{0+}^{\gamma-2}(T_1x)(\tau_2)|\to0$。此时 $|D_{0+}^{\gamma-1}(T_1x)(\tau_1)-D_{0+}^{\gamma-1}(T_1x)(\tau_2)|=0$,故当 $\tau_1\to\tau_2$ 时,$|D_{0+}^{\gamma-1}(T_1x)(\tau_1)-D_{0+}^{\gamma-1}(T_1x)(\tau_2)|\to0$。所以 T_1 在 B_r 上等度连续。应用 Arzela-Ascoli 定理可得 T_1 在 B_r 上是紧的,因此,T_1 是全连续算子。根据条件(H_3),同定理 1 的证明,可得 T_2 是压缩算子。因此,由引理 3 可知问题(1) 至少存在 1 个解。

3　应用举例

例 1　考虑下列分数阶脉冲微分方程边值问题:

$$\begin{cases}D_{0+}^{\frac{5}{2},\frac{1}{2}}x(t)=\dfrac{t|x(t)|}{10(1+|x(t)|)}+\dfrac{t^2\sin|D_{0+}^{\frac{3}{4}}x(t)|}{20}+\dfrac{t^3\cos|D_{0+}^{\frac{7}{4}}x(t)|}{30},\quad t\in J,\quad t\neq\dfrac{1}{2},\\ \Delta x(\frac{1}{2})=\frac{1}{50}x(\frac{1}{2}),\quad \Delta D_{0+}^{\frac{3}{4}}x(\frac{1}{2})=\frac{1}{100}x(\frac{1}{2}),\quad \Delta D_{0+}^{\frac{7}{4}}x(\frac{1}{2})=\frac{1}{200}x(\frac{1}{2}),\\ x(0)=0,D_{0+}^{\frac{3}{4}}x(0)=D_{0+}^{\frac{7}{4}}x(1),\ D_{0+}^{\frac{3}{4}}x(1)=100D_{0+}^{\frac{7}{4}}x(0),\end{cases}\tag{6}$$

对应问题(1):$\alpha=\frac{5}{2},\beta=\frac{1}{2},\gamma=\frac{11}{4},m=1,k=1,a=1,b=100$,则 $b>a+1,I_1(x)=\frac{1}{50}x,\bar{I}_1(x)=\frac{1}{100}x$,$\tilde{I}_1(x)=\frac{1}{200}x$。显然,$f,I_1,\bar{I}_1,\tilde{I}_1$ 在 J 上连续,且满足 $(1-t)^{-\frac{1}{4}}\left(\frac{t}{10}+\frac{t^2}{20}+\frac{t^3}{30}\right)\in L[0,1]$,对 $\forall t\in J,x,y,z\in\Omega_9=\{(x,y,z)\mid\max\{|x|,|y|,|z|\}\leqslant9\}$,有 $|f(t,x,y,z)|=\frac{t|x(t)|}{10(1+|x(t)|)}+\frac{t^2\sin|y(t)|}{20}+\frac{t^3\cos|z(t)|}{30}\leqslant\frac{t}{10}+\frac{t^2}{20}+\frac{t^3}{30}=h(t)$,且 $H_9\approx0.1187\leqslant3$;对 $\forall t\in J,x\in\boldsymbol{R}$,$|I_1(x)|=\frac{1}{50}|x|\leqslant0.1|x|=c_1|x|$,$|\bar{I}_1(x)|=\frac{1}{100}|x|\leqslant0.02|x|=d_1|x|$,$|\tilde{I}_1(x)|=\frac{1}{200}|x|\leqslant0.01|x|=e_1|x|$。存在 $K_1=\frac{1}{10}>0,K_2=\frac{1}{20}>0,K_3=\frac{1}{30}>0$,对 $\forall t\in J,x,\bar{x},y,\bar{y},z,\bar{z}\in R$,有 $|f(t,x,$

$y,z)-f(t,\bar{x},\bar{y},\bar{z})|\leqslant\frac{1}{10}|x(t)-\bar{x}(t)|+\frac{1}{20}|y(t)-\bar{y}(t)|+\frac{1}{30}|z(t)-\bar{z}(t)|=K_1|x-\bar{x}|+K_2|y-\bar{y}|+K_3|z-\bar{z}|$。经计算，$m\approx 0.199\ 5<1$，$M_1\approx 0.697\ 4$，$M_2\approx 0.688\ 5$，故 $M=\max\{M_1,M_2\}=0.697\ 4\leqslant 1$。因此，根据定理1可知，边值问题(6)至少有1个解。

例2 考虑下列分数阶脉冲微分方程边值问题：

$$\begin{cases}D_{0+}^{\frac{7}{3},\frac{1}{3}}x(t)=f(t,x(t),D_{0+}^{\frac{5}{9}}x(t),D_{0+}^{\frac{14}{9}}x(t)),\quad t\in J,t\neq\frac{1}{3},\\ \Delta x(\frac{1}{3})=\frac{1}{30}x(\frac{1}{2}),\quad \Delta D_{0+}^{\frac{5}{9}}x(\frac{1}{3})=\frac{1}{40}x(\frac{1}{3}),\quad \Delta D_{0+}^{\frac{14}{9}}x(\frac{1}{3})=\frac{1}{50}x(\frac{1}{3}),\\ x(0)=0,\ D_{0+}^{\frac{5}{9}}x(0)=2D_{0+}^{\frac{14}{9}}x(1),\ D_{0+}^{\frac{5}{9}}x(1)=200D_{0+}^{\frac{14}{9}}x(0),\end{cases}\tag{7}$$

其中 $f(t,x(t),D_{0+}^{\frac{5}{9}}x(t),D_{0+}^{\frac{14}{9}}x(t))=\frac{t^4\sin|x(t)|}{10}+\frac{t^5\cos|D_{0+}^{\frac{5}{9}}x(t)|}{5}+\frac{3t^6|D_{0+}^{\frac{14}{9}}x(t)|}{10(1+|D_{0+}^{\frac{14}{9}}x(t)|)}$。这里 $\alpha=\frac{7}{3}$，$\beta=\frac{1}{3}$，$\gamma=\frac{23}{9}$，$m=1$，$k=1$，$a=2$，$b=200$，$I_1(x)=\frac{1}{30}x$，$\bar{I}_1(x)=\frac{1}{40}x$，$\tilde{I}_1(x)=\frac{1}{50}x$，$f\in C(J\times R^3,R)$，$I_1,\bar{I}_1,\tilde{I}_1\in C(R,R)$，且满足 $(1-t)^{-\frac{2}{9}}(\frac{t^4}{10}+\frac{t^5}{5}+\frac{3t^6}{10})\in L[0,1]$，对 $\forall t\in J$，$x,y,z\in R$，有 $|f(t,x,y,z)|\leqslant\frac{t^4}{10}+\frac{t^5}{5}+\frac{3t^6}{10}=g(t)$，$G\approx 0.063\ 2$；对 $\forall t\in J$，$x\in R$，有 $|I_1(x)|=\frac{1}{30}|x|\leqslant\frac{1}{20}|x|=c_1|x|$，$|\bar{I}_1(x)|=\frac{1}{40}|x|\leqslant\frac{1}{30}|x|=d_1|x|$，$|\tilde{I}_1(x)|=\frac{1}{50}|x|\leqslant\frac{1}{40}|x|=e_1|x|$；存在 $K_1=\frac{1}{10}>0$，$K_2=\frac{1}{5}>0$，$K_3=\frac{3}{10}>0$，对 $\forall t\in J$，$x,\bar{x}.y,\bar{y},z,\bar{z}\in R$，有 $|f(t,x,y,z)-f(t,\bar{x},\bar{y},\bar{z})|\leqslant\frac{t^4|\sin|x(t)|-\sin|\bar{x}(t)||}{10}+\frac{t^5|\cos|y(t)-\cos|\bar{y}(t)||}{5}+\frac{3t^6||z(t)|-|\bar{z}(t)||}{10(1+|z(t)|)(1+|\bar{z}(t)|}\leqslant\frac{1}{10}|x(t)-\bar{x}(t)|+\frac{1}{5}|y(t)-\bar{y}(t)|+\frac{3}{10}|z(t)-\bar{z}(t)|=K_1|x-\bar{x}|+K_2|y-\bar{y}|+K_3|z-\bar{z}|$。另外，经计算，得 $m\approx 0.648\ 2<1$，$\rho_1\approx 0.163\ 4$，$\rho_2\approx 0.110\ 0$，所以 $\rho=\max\{\rho_1,\rho_2\}=0.163\ 4<1$，故满足定理2的所有条件，因此，边值问题(7)至少存在1个解。

4 结 语

本文运用Krasnoselskii不动点定理，研究了一类具有Hilfer分数阶导数的脉冲微分方程边值问题，得到了这类边值问题解的存在性，通过2个具体实例说明了结论的正确性。研究结果表明，Krasnoselskii不动点定理能够有效解决具有Hilfer分数阶脉冲微分方程边值问题解的存在性问题，推广了具有Riemann-Liouville分数阶导数或者Caputo分数阶导数脉冲微分方程边值问题的结果，丰富了分数阶微分方程理论，为解决其他类型的脉冲分数阶微分方程边值问题提供了借鉴与参考。

但是本文是在非线性项连续的条件下考虑的脉冲边值问题，限制条件较强。在今后的研究中，将会探索削弱非线性项满足的条件，进一步研究此类问题存在解的更一般化的结果。

参考文献/References：

[1] TIAN Yu，GE Weigao. Applications of variational methods to boundary-value problem for impulsive differential equations[J]. Proceedings of the Edinburgh Mathematical Society，2008，51(2)：509-527.

[2] NIETO J J，O'REGAN D. Variational approach to impulsive differential equations[J]. Nonlinear Analysis：Real World Applications，2009，10(2)：680-690.

[3] ZHANG Hao，LI Zhixiang. Variational approach to impulsive differential equations with periodic boundary conditions[J]. Nonlinear Analysis：Real World Applications，2010，11(1)：67-78.

[4] SAKTHIVEL R，ANANDHI E R. Approximate controllability of impulsive differential equations with state-dependent delay[J]. International Journal of Control，2010，83(2)：387-393.

[5] XIAO J,NIETO J J. Variational approach to some damped Dirichlet nonlinear impulsive differential equations[J]. Journal of the Franklin Institute,2011,348(2):369-377.

[6] FEČKAN M,ZHOU Y,WANG J R. On the concept and existence of solution for impulsive fractional differential equations[J]. Communications in Nonlinear Science and Numerical Simulation,2012,17(7):3050-3060.

[7] WANG J R,ZHOU Y,FEČKAN M. On recent developments in the theory of boundary value problems for impulsive fractional differential equations[J]. Computers & Mathematics with Applications,2012,64(10):3008-3020.

[8] STAMOVA I M. Mittag-Leffler stability of impulsive differential equations of fractional order[J]. Quarterly of Applied Mathematics, 2015,73(3):525-535.

[9] 王晓君,薛琳博,王彦朋. 基于 STFRFT 的脉冲干扰抑制方法研究[J]. 河北科技大学学报,2021,42(1):15-21.
WANG Xiaojun,XUE Linbo,WANG Yanpeng. Research on impulse interference suppression method based on short time fractional Fourier transform[J]. Journal of Hebei University of Science and Technology,2021,42(1):15-21.

[10] FENG Meiqiang,XIE Dongxiu. Multiple positive solutions of multi-point boundary value problem for second-order impulsive differential equations[J]. Journal of Computational and Applied Mathematics,2009,223(1):438-448.

[11] LIU Z H,LU L,SZANTO I. Existence of solutions for fractional impulsive differential equations with p-Laplacian operator[J]. Acta Mathematica Hungarica,2013,141(3):203-219.

[12] LIU Yansheng. Bifurcation techniques for a class of boundary value problems of fractional impulsive differential equations[J]. Journal of Nonlinear Sciences and Applications,2015,8(4):340-353.

[13] WANG J R,FECKAN M. A survey on impulsive fractional differential equations[J]. Fractional Calculus and Applied Analysis,2016,19(4):806-831.

[14] YAN Zuomao,LU Fangxia. Approximate controllability of a multi-valued fractional impulsive stochastic partial integro-differential equation with infinite delay[J]. Applied Mathematics and Computation,2017,292:425-447.

[15] DUAN Lijing,XIE Jingli. Existence results for the boundary value problems of fractional impulsive differential equations with p-Laplacian operator[J]. Applied Mathematical Sciences,2021,15(3):101-111.

[16] 江卫华,李庆敏,周彩莲. 分数阶脉冲微分方程边值问题解的存在性[J]. 河北科技大学学报,2016,37(6):562-574.
JIANG Weihua,LI Qingmin,ZHOU Cailian. Existence of solutions to boundary value problem of fractional differential equations with impulsive[J]. Journal of Hebei University of Science and Technology,2016,37(6):562-574.

[17] SUBASHINI R,JOTHIMANI K,NISAR K S,et al. New results on nonlocal functional integro-differential equations via Hilfer fractional derivative[J]. Alexandria Engineering Journal,2020,59(5):2891-2899.

[18] REDHWAN S,SHAIKH S L. Implicit fractional differential equation with nonlocal integral-multipoint boundary conditions in the frame of Hilfer fractional derivative[J]. Journal of Mathematical Analysis and Modeling,2021,2(1):62-71.

[19] KAVITHA K,VIJAYAKUMAR V,UDHAYAKUMAR R,et al. Results on controllability of Hilfer fractional differential equations with infinite delay via measures of noncompactness[J]. Asian Journal of Control,2022,24(3):1406-1415.

[20] HILFER R. Applications of Fractional Calculus in Physics[M]. Singapore:World Scientific,2000.

[21] KILBAS A A,SRIVASTAVA H M,TRUJILLO J J. Theory and Applications of Fractional Differential Equations[M]. Amsterdam: Elsevier,2006.

[22] ZHOU Yong. Basic Theory of Fractional Differential Equations[M]. Hackensack:World Scientific,2014.