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分数阶脉冲微分方程组边值问题解的存在性

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摘　要：通过定义合适的线性空间以及范数，给出恰当的算子，在非线性项和脉冲值满足一定的条件下，分别利用压缩映像原理和krasnoselskii不动点定理，研究了分数阶脉冲微分方程组边值问题解的存在性和唯一性，并给出例子说明所需要的条件是可以满足的。

关键词：常微分方程数值解；压缩映像原理；微分方程组；脉冲；分数阶微积分；边值问题

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Existence of solutions for boundary value problem of fractional order impulsive differential equations systems

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Abstract: By defining appropriate linear space and norm, giving the appropriate operator, using the contraction mapping principle and krasnoselskii fixed point theorem respectively, the existence and uniqueness of solutions for boundary value problem of fractional order impulsive differential equations systems are investigated under certain condition that nonlinear term and pulse value are satisfied. An example is given to illustrate that the required conditions can be satisfied.

Keywords: numerical solution of ordinary differential equations; contraction mapping principle; differential equations; impulsive; fractional calculus; boundary value problem

1　问题提出

分数阶微分方程的出现已有300多年历史，由于其理论的自身发展以及在流变学、电子网络、黏弹性、物理化学、分析化学、生物学和控制理论等学科中的广泛应用，使得分数阶微分方程受到越来越多学者的广泛关注[1-21]。

文献[1]研究了非线性脉冲微分方程边值问题：

$$\begin{cases} {}^{c}D^{q}x(t)=f(t,x(t)), \quad 1<q\leqslant 2, \\ \Delta x(t_k)=I_k(x(t_k^-)), \quad \Delta x'(t_k)=J_k(u(t_k^-)), k=1,2,\cdots,p, \\ x(0)+x'(0)=0, \quad x(1)+x'(1)=0 \end{cases}$$

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解的存在性。

受上述文献的启发,本文研究如下非线性脉冲微分方程组边值问题:

$$\begin{cases} {}^cD^qu(t)=f_1(t,u,v), \quad 1<q\leqslant 2, t\in J', \\ {}^cD^pv(t)=f_2(t,u,v), \quad 1<p\leqslant 2, t\in J', \\ \Delta u(t_k)=I_{1,k}(u(t_k^-)), \quad \Delta v(t_k)=I_{2,k}(v(t_k^-)), k=1,2,\cdots,n, \\ \Delta u'(t_k)=J_{1,k}(u(t_k^-)), \quad \Delta v'(t_k)=J_{2,k}(v(t_k^-)), k=1,2,\cdots,n, \\ u(0)+u'(0)=0, \quad v(0)+v'(0)=0, \\ u(1)+u'(1)=0, \quad v(1)+v'(1)=0 \end{cases} \tag{1}$$

解的存在性。其中

$J'_1=(0,1)$,$J'=J'_1\backslash\{t_1,t_2,\cdots,t_n\}$,$J=[0,1]$,$0=t_0<t_1<t_2<\cdots<t_n<t_{n+1}=1$,$k=1,2,\cdots,n$,$f_i\in C(J\times R\times R,R)$,$I_{i,k}\in C(R,R)$,$J_{i,k}\in C(R,R)$,$(i=1,2)$,$\Delta u(t)=u(t_k^+)-u(t_k^-)$,$\Delta v(t)=v(t_k^+)-v(t_k^-)$,$\Delta u'(t)=u'(t_k^+)-u'(t_k^-)$,$\Delta v'(t)=v'(t_k^+)-v'(t_k^-)$,$u(t_k^+)=\lim_{h\to 0^+}u(t_k+h)$,$u(t_k^-)=\lim_{h\to 0^-}u(t_k+h)=u(t_k)$,$v(t_k^+)=\lim_{h\to 0^+}v(t_k+h)$,$v(t_k^-)=\lim_{h\to 0^-}v(t_k+h)=v(t_k)$。

2　预备知识

定义 1[1]　函数 u 定义在区间$(0,1)$,$q>0$,则阶数为 q 的 Riemann-Liouville 分数阶积分定义为

$$I^qu(t)=\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}u(s)\mathrm{d}s, \quad t\in[0,1]。$$

定义 2[1]　函数 u 定义在区间$(0,1)$,$q>0$,则阶数为 q 的 Riemann-Liouville 分数阶导数定义为

$$D^qu(t)=\frac{1}{\Gamma(n-q)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_0^t(t-s)^{n-q-1}u(s)\mathrm{d}s, \quad n-1<q<n, t\in[0,1]。$$

取空间

$PC(J,R)=\{x:J\to R;x\in C((t_k,t_{k+1}],R),k=0,1,2,\cdots,n,x(t_k^-)=x(t_k),k=1,2,\cdots,n\}$,范数为 $\|x\|_{PC}=\sup_{t\in J}|x(t)|$。

$PC^1(J,R)=\{x'\in PC(J,R);x'(t_k^-)=x'(t_k),k=1,2,\cdots,n)$,范数为 $\|x\|_{PC^1}=\max\{\|x\|_{PC},\|x'\|_{PC}\}$。

由文献[1]知 $PC(J,R)$ 及 $PC^1(J,R)$ 是 Banach 空间。乘积空间 $PC^1(J,R)\times PC^1(J,R)$,范数定义为 $\|(u,v)\|=\max\{\|u\|_{PC^1},\|v\|_{PC^1}\}$,$\forall(u,v)\in PC^1(J,R)\times PC^1(J,R)$,是一个 Banach 空间。

定理 1[1]　(压缩映像原理)设 X 是完备的度量空间,T 是 X 上的压缩映像,那么 T 有且只有一个不动点。

定理 2[1]　(krasnoselskii 不动点定理)设 M 是 Banach 空间 X 中的一个非空凸子集。假设 A,B 是 2 个映射,使得:

1) 对任意的 $x,y\in M$,有 $Ax+By\in M$;

2) A 是全连续映射;

3) B 是一个压缩映射,

则存在至少一个 $z\in M$,使得 $z=Az+Bz$。

引理 1[1]　假设

$$J_k=\begin{cases}[t_0,t_1], & k=0, \\ (t_k,t_{k+1}], & k=1,2,\cdots,n\end{cases} \quad 和 \quad \chi(t)=\begin{cases}0, & t\in J_0, \\ 1, & t\notin J_0。\end{cases}$$

对于 $\varphi,\xi\in C[0,1]$ 则分数阶脉冲微分方程组边值问题:

$$\begin{cases} {}^cD^qu(t)=\varphi(t), \quad 1<q\leqslant 2, t\in J', \\ {}^cD^pv(t)=\xi(t), \quad 1<p\leqslant 2, t\in J', \\ \Delta u(t_k)=I_{1,k}(u(t_k^-)), \quad \Delta v(t_k)=I_{2,k}(v(t_k^-)), \quad k=1,2,\cdots,n, \\ \Delta u'(t_k)=J_{1,k}(u(t_k^-)), \quad \Delta v'(t_k)=J_{2,k}(v(t_k^-)), \quad k=1,2,\cdots,n, \\ u(0)+u'(0)=0, \quad v(0)+v'(0)=0, \\ u(1)+u'(1)=0, \quad v(1)+v'(1)=0 \end{cases} \tag{2}$$

的解为$(u(t),v(t))$。其中

$$u(t)=\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\varphi(s)\mathrm{d}s+(1-t)\Big[\int_{t_n}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}\varphi(s)\mathrm{d}s+\int_{t_n}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}\varphi(s)\mathrm{d}s+$$

$$\sum_{0<t_k<1}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}\varphi(s)\mathrm{d}s+I_{1,k}(u(t_k^-))\Big)+\sum_{0<t_k<1}(2-t_k)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\varphi(s)\mathrm{d}s+J_{1,k}(u(t_k^-))\Big)\Big]+$$

$$\chi(t)\sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}\varphi(s)\mathrm{d}s+I_{1,k}(u(t_k^-))\Big)+$$

$$\chi(t)\sum_{0<t_k<1}(t-t_k)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\varphi(s)\mathrm{d}s+J_{1,k}(u(t_k^-))\Big)。$$

$$v(t)=\frac{1}{\Gamma(p)}\int_{t_k}^{t}(t-s)^{p-1}\zeta(s)\mathrm{d}s+(1-t)\Big[\int_{t_n}^{1}\frac{(1-s)^{p-1}}{\Gamma(p)}\zeta(s)\mathrm{d}s+\int_{t_n}^{1}\frac{(1-s)^{p-2}}{\Gamma(p-1)}\zeta(s)\mathrm{d}s+$$

$$\sum_{0<t_k<1}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{p-1}}{\Gamma(p)}\zeta(s)\mathrm{d}s+I_{2,k}(v(t_k^-))\Big)+\sum_{0<t_k<1}(2-t_k)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{p-2}}{\Gamma(p-1)}\zeta(s)\mathrm{d}s+J_{2,k}(v(t_k^-))\Big)\Big]+$$

$$\chi(t)\sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{p-1}}{\Gamma(p)}\zeta(s)\mathrm{d}s+I_{2,k}(v(t_k^-))\Big)+$$

$$\chi(t)\sum_{0<t_k<1}(t-t_k)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{p-2}}{\Gamma(p-1)}\zeta(s)\mathrm{d}s+J_{2,k}(v(t_k^-))\Big)。$$

3 主要结果

定理 3 令 $f_i\in C(J\times R\times R,R)$，$I_{i,k}\in C(R,R)$，$J_{i,k}\in C(R,R)$，$(i=1,2)$，$f_i$ 是连续有界函数，$I_{i,k}$，$J_{i,k}$ 是连续函数。假如存在正实数 $L_1,L_2,L_3,M_2,M_3,l_1,l_2,l_3,m_2,m_3$ 使得：

H_1） $|f_1(t,u_1,v_1)-f_1(t,u_2,v_2)|\leqslant L_1\max\{|u_1-u_2|,|v_1-v_2|\},\forall t\in[0,1]$，

$|f_2(t,u_1,v_1)-f_2(t,u_2,v_2)|\leqslant l_1\max\{|u_1-u_2|,|v_1-v_2|\},\forall t\in[0,1]$，

H_2） $|I_{1,k}(u_1)-I_{1,k}(u_2)|\leqslant L_2|u_1-u_2|$，$|J_{1,k}(u_1)-J_{1,k}(u_2)|\leqslant L_3|u_1-u_2|$，$|I_{1,k}(u)|\leqslant M_2$，$|J_{1,k}(u)|\leqslant M_3$，$|I_{2,k}(v_1)-I_{2,k}(v_2)|\leqslant l_2|v_1-v_2|$，$|J_{2,k}(v_1)-J_{2,k}(v_2)|\leqslant l_3|v_1-v_2|$，$|I_{2,k}(v)|\leqslant m_2$，$|J_{2,k}(v)|\leqslant m_3$，$k=1,2,\cdots,n$。

H_3） $\max\{\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4\}<1$，

其中 $\Lambda_1=L_1[\frac{2(1+n)}{\Gamma(q+1)}+\frac{1+3n}{\Gamma(q)}]+n(2L_2+3L_3)$， $\Lambda_2=l_1[\frac{2(1+n)}{\Gamma(p+1)}+\frac{1+3n}{\Gamma(p)}]+n(2l_2+3l_3)$，

$$\Lambda_3=L_1(\frac{1+n}{\Gamma(q+1)}+\frac{2+3n}{\Gamma(q)})+n(L_2+3L_3),\quad \Lambda_4=l_1(\frac{1+n}{\Gamma(p+1)}+\frac{2+3n}{\Gamma(p)})+n(l_2+3l_3)。$$

$$L_1\leqslant\max\{\frac{1}{2}\Big[(\frac{2(1+n)}{\Gamma(q+1)}+\frac{1+3n}{\Gamma(q)})\Big]^{-1},\quad \frac{1}{2}\Big[\frac{1+n}{\Gamma(q+1)}+\frac{2+3n}{\Gamma(q)}\Big]^{-1}\},$$

$$l_1\leqslant\max\{\frac{1}{2}\Big[(\frac{2(1+n)}{\Gamma(p+1)}+\frac{1+3n}{\Gamma(p)})\Big]^{-1},\quad \frac{1}{2}\Big[\frac{1+n}{\Gamma(p+1)}+\frac{2+3n}{\Gamma(p)}\Big]^{-1}\}$$

成立，则分数阶脉冲微分方程组(1) 在 $PC^1(J,R)\times PC^1(J,R)$ 中有唯一解。

证明 定义映射 $A:PC^1(J,R)\times PC^1(J,R)\rightarrow PC^1(J,R)\times PC^1(J,R)$ 如下：

$$A(u,v)(t)=(X(u,v)(t),Y(u,v)(t)),$$

其中：

$$X(u,v)(t)=(1-t)\Big[\int_{t_n}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}f_1(s,u(s),v(s))\mathrm{d}s+$$

$$\int_{t_n}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}f_1(s,u(s),v(s))\mathrm{d}s+$$

$$\sum_{0<t_k<1}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}f_1(s,u(s),v(s))\mathrm{d}s+I_{1,k}(u(t_k^-))\Big)+$$

$$\sum_{0<t_k<1}(2-t_k)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f_1(s,u(s),v(s))\mathrm{d}s+J_{1,k}(u(t_k^-))\Big)\Big]+$$

$$\chi(t)\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}f_1(s,u(s),v(s))\mathrm{d}s+$$

$$\chi(t)\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f_1(s,u(s),v(s))\mathrm{d}s+$$

$$\int_{t_k}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}f_1(s,u(s),v(s))\mathrm{d}s+$$

$$\chi(t)\sum_{0<t_k<t}I_{1,k}(u(t_k^-))+$$

$$\chi(t)\sum_{0<t_k<t}(t-t_k)J_{1,k}(u(t_k^-)),$$

$$Y(u,v)(t)=(1-t)\Big[\int_{t_n}^{1}\frac{(1-s)^{p-1}}{\Gamma(p)}f_2(s,u(s),v(s))\mathrm{d}s+$$

$$\int_{t_n}^{1}\frac{(1-s)^{p-2}}{\Gamma(p-1)}f_2(s,u(s),v(s))\mathrm{d}s+$$

$$\sum_{0<t_k<1}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{p-1}}{\Gamma(p)}f_2(s,u(s),v(s))\mathrm{d}s+I_{2,k}(v(t_n^-))\Big)+$$

$$\sum_{0<t_n<1}(2-t_n)\Big(\int_{t_{n-1}}^{t_n}\frac{(t_n-s)^{p-2}}{\Gamma(p-1)}f_2(s,u(s),v(s))\mathrm{d}s+J_{2,n}(v(t_n^-))\Big)\Big]+$$

$$\chi(t)\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{p-1}}{\Gamma(p)}f_2(s,u(s),v(s))\mathrm{d}s+$$

$$\chi(t)\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{p-2}}{\Gamma(p-1)}f_2(s,u(s),v(s))\mathrm{d}s+$$

$$\int_{t_k}^{t}\frac{(t-s)^{p-1}}{\Gamma(p)}f_2(s,u(s),v(s))\mathrm{d}s+$$

$$\chi(t)\sum_{0<t_k<t}I_{2,k}(v(t_k^-))+$$

$$\chi(t)\sum_{0<t_k<t}(t-t_k)J_{2,k}(v(t_k^-))。$$

$$A'(u,v)(t)=(X'(u,v)(t),Y'(u,v)(t)),$$

其中：

$$X'(u,v)(t)=\int_{t_k}^{t}\frac{(t-s)^{q-2}}{\Gamma(q-1)}f_1(s,u(s),v(s))\mathrm{d}s+$$

$$\chi(t)\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f_1(s,u(s),v(s))\mathrm{d}s+$$

$$\chi(t)\sum_{0<t_k<t}J_{1,k}(u(t_k^-))-\Big[\int_{t_n}^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}f_1(s,u(s),v(s))\mathrm{d}s+$$

$$\int_{t_n}^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}f_1(s,u(s),v(s))\mathrm{d}s+$$

$$\sum_{0<t_k<1}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}f_1(s,u(s),v(s))\mathrm{d}s+I_{1,k}(u(t_k^-))\Big)+$$

$$\sum_{0<t_n<1}(2-t_k)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f_1(s,u(s),v(s))\mathrm{d}s+J_{1,k}(u(t_k^-))\Big)\Big],$$

$$Y'(u,v)(t)=\int_{t_k}^{t}\frac{(t-s)^{p-2}}{\Gamma(p-1)}f_2(s,u(s),v(s))\mathrm{d}s+$$

$$\chi(t)\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{p-2}}{\Gamma(p-1)}f_2(s,u(s),v(s))\mathrm{d}s+$$

$$\chi(t)\sum_{0<t_k<t}J_{2,k}(v(t_k^-))-\Big[\int_{t_n}^{1}\frac{(1-s)^{p-1}}{\Gamma(p)}f_2(s,u(s),v(s))\mathrm{d}s+$$

$$\int_{t_n}^{1}\frac{(1-s)^{p-2}}{\Gamma(p-1)}f_2(s,u(s),v(s))\mathrm{d}s+$$

$$\sum_{0<t_k<1}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{p-1}}{\Gamma(p)}f_2(s,u(s),v(s))\mathrm{d}s+I_{2,k}(v(t_k^-))\Big)+$$

$$\sum_{0<t_k<1}(2-t_k)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{p-2}}{\Gamma(p-1)}f_2(s,u(s),v(s))\mathrm{d}s+J_{2,k}(v(t_k^-))\Big)\Big]。$$

定义 $\sup_{t\in[0,1]}|f_1(t,0,0)|=M_1$，$\sup_{t\in[0,1]}|f_2(t,0,0)|=m_1$。取

$$r_1\geqslant 2[M_1(\frac{2(1+n)}{\Gamma(q+1)}+\frac{1+3n}{\Gamma(q)})+n(2M_2+3M_3)],\quad r_2\geqslant 2[m_1(\frac{2(1+n)}{\Gamma(p+1)}+\frac{1+3n}{\Gamma(p)})+n(2m_2+3m_3)]。$$

$$r_3\geqslant 2[M_1(\frac{1+n}{\Gamma(q+1)}+\frac{2+3n}{\Gamma(q)})+n(M_2+3M_3)],\quad r_4\geqslant 2[m_1(\frac{1+n}{\Gamma(p+1)}+\frac{2+3n}{\Gamma(p)})+n(m_2+3m_3)]。$$

$|f_1(s,u(s),v(s))|\leqslant|f_1(s,u(s),v(s))-f_1(s,0,0)|+|f_1(s,0,0)|\leqslant L_1\max\{|u(s)|,|v(s)|\}+M_1\leqslant L_1\max\{\|u\|_{PC},\|v\|_{PC}\}+M_1\leqslant L_1\max\{\|u\|_{PC^1},\|v\|_{PC^1}\}+M_1=L_1\|(u,v)\|+M_1\leqslant L_1 r+M_1$，

其中 $r=\max\{r_1,r_2,r_3,r_4\}$，

下面需要证明 $A(B_r)\subset B_r$，其中 $B_r=\{(u,v)\in PC^1(J,R)\times PC^1(J,R):\|(u,v)\|_{PC^1}\leqslant r\}$。

对于$(u,v)\in B_r$ 有

$$\|X(u,v)\|_{PC}\leqslant\int_{t_n}^1\frac{(1-s)^{q-1}}{\Gamma(q)}|f_1(s,u(s),v(s))|\mathrm{d}s+\int_{t_n}^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}|f_1(s,u(s),v(s))|\mathrm{d}s+$$
$$\sum_{0<t_k<1}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}|f_1(s,u(s),v(s))|\mathrm{d}s+|I_{1,k}(u(t_k^-))|\Big)+$$
$$\sum_{0<t_k<1}(2-t_k)\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}|f_1(s,u(s),v(s))|\mathrm{d}s+|J_{1,k}(u(t_k^-))|\Big)+$$
$$\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}|f_1(s,u(s),v(s))|\mathrm{d}s+\sum_{0<t_k<t}|(t-t_k)|\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}|f_1(s,u(s),v(s))|\mathrm{d}s+$$
$$\int_{t_k}^t\frac{(t-s)^{q-1}}{\Gamma(q)}|f_1(s,u(s),v(s))|\mathrm{d}s+\sum_{0<t_k<t}|I_{1,k}(u(t_k^-))|+\sum_{0<t_k<t}|(t-t_k)||J_{1,k}(u(t_k^-))|\leqslant$$
$$L_1[\frac{2(1+n)}{\Gamma(q+1)}+\frac{1+3n}{\Gamma(q)}]r+[M_1(\frac{2(1+n)}{\Gamma(q+1)}+\frac{1+3n}{\Gamma(q)})+n(2M_2+3M_3)]\leqslant\frac{r_1+r}{2}\leqslant r。$$

类似可得：

$$\|Y(u,v)\|_{PC}\leqslant l_1[\frac{2(1+n)}{\Gamma(p+1)}+\frac{1+3n}{\Gamma(p)}]r+[m_1(\frac{2(1+n)}{\Gamma(p+1)}+\frac{1+3n}{\Gamma(p)})+n(2m_2+3m_3)]\leqslant\frac{r_2+r}{2}\leqslant r,$$

$$\|X'(u,v)\|_{PC}\leqslant L_1(\frac{1+n}{\Gamma(q+1)}+\frac{1+3n}{\Gamma(q)}]r+[M_1(\frac{1+n}{\Gamma(q+1)}+\frac{2+3n}{\Gamma(q)})+n(M_2+3M_3)]\leqslant\frac{r_3+r}{2}\leqslant r,$$

$$\|Y'(u,v)\|_{PC}\leqslant l_1(\frac{1+n}{\Gamma(p+1)}+\frac{2+3n}{\Gamma(p)}]r+[m_1(\frac{1+n}{\Gamma(p+1)}+\frac{2+3n}{\Gamma(p)})+n(m_2+3m_3)]\leqslant\frac{r_4+r}{2}\leqslant r。$$

所以有：

$\|A(u,v)\|_{PC^1}=\max\{\|X(u,v)\|_{PC^1},\|Y(u,v)\|_{PC^1}\}=\max\{\max\{\|X(u,v)\|_{PC},\|X'(u,v)\|_{PC}\},\max\{\|Y(u,v)\|_{PC},\|Y'(u,v)\|_{PC}\}\}\leqslant r$，即 $A(B_r)\subset B_r$。

再证 $A:B_r\rightarrow B_r$ 是压缩映像。

$$\begin{aligned}&|f_1(s,u_1(s),v_1(s))-f_1(s,u_2(s),v_2(s))|\leqslant\\&L_1\max\{|u_1(s)-u_2(s)|,|v_1(s)-v_2(s)|\}\leqslant\\&L_1\max\{\|u_1-u_2\|_{PC},\|v_1-v_2\|_{PC}\}\leqslant\\&L_1\max\{\|u_1-u_2\|_{PC^1},\|v_1-v_2\|_{PC^1}\}=\\&L_1\|(u_1-u_2,v_1-v_2)\|_{PC^1}=\\&L_1\|(u_1,v_1)-(u_2,v_2)\|_{PC^1},\end{aligned}$$

$$\begin{aligned}&|I_{1,k}(u_1(t_k^-))-I_{1,k}(u_2(t_k^-))|\leqslant\\&L_2|u_1(t_k^-)-u_2(t_k^-)|\leqslant\\&L_2\|u_1-u_2\|_{PC}\leqslant\\&L_2\max\{\|u_1-u_2\|_{PC},\|v_1-v_2\|_{PC}\}\leqslant\\&L_2\max\{\|u_1-u_2\|_{PC^1},\|v_1-v_2\|_{PC}\}=\\&L_2\|(u_1-u_2,v_1-v_2)\|_{PC^1}=\end{aligned}$$

$$
\begin{aligned}
&L_2 \| (u_1, v_1) - (u_2, v_2) \|_{PC^1},\\
&| J_{1,k}(u_1(t_k^-)) - J_{1,k}(u_2(t_k^-)) | \leqslant\\
&L_3 \, | u_1(t_k^-) - u_2(t_k^-) | \leqslant\\
&L_3 \| u_1 - u_2 \|_{PC} \leqslant\\
&L_3 \max\{ \| u_1 - u_2 \|_{PC}, \| v_1 - v_2 \|_{PC} \} \leqslant\\
&L_3 \max\{ \| u_1 - u_2 \|_{PC^1}, \| v_1 - v_2 \|_{PC^1} \} =\\
&L_3 \| (u_1 - u_2, v_1 - v_2) \|_{PC^1} =\\
&L_3 \| (u_1, v_1) - (u_2, v_2) \|_{PC^1}。
\end{aligned}
$$

对于$(u_1, v_1), (u_2, v_2) \in PC^1(J, R) \times PC^1(J, R), \forall t \in [0, 1]$，有：

$$
\begin{aligned}
&\| X(u_1, v_1) - X(u_2, v_2) \|_{PC} \leqslant \int_{t_n}^{t} \frac{(1-s)^{q-1}}{\Gamma(q)} \, | f_1(s, u_1(s), v_1(s)) - f_1(s, u_2(s), v_2(s)) | \, \mathrm{d}s +\\
&\int_{t_n}^{1} \frac{(1-s)^{q-2}}{\Gamma(q-1)} \, | f_1(s, u_1(s), v_1(s)) - f_1(s, u_2(s), v_2(s)) | \, \mathrm{d}s +\\
&\sum_{0<t_k<1} \left(\int_{t_{k-1}}^{t_k} \frac{(t_k - s)^{q-1}}{\Gamma(q)} \, | f_1(s, u_1(s), v_1(s)) - f_1(s, u_2(s), v_2(s)) | \, \mathrm{d}s + | I_{1,k}(u_1(t_k^-)) - I_{1,k}(u_2(t_k^-)) | \right) +\\
&\sum_{0<t_k<1} (2 - t_k) \left(\int_{t_{k-1}}^{t_k} \frac{(t_k - s)^{q-2}}{\Gamma(q-1)} \, | f_1(s, u_1(s), v_1(s)) - f_1(s, u_2(s), v_2(s)) | \, \mathrm{d}s + | J_{1,k}(u_1(t_k^-)) - J_{1,k}(u_2(t_k^-)) | \right) +\\
&\sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} \frac{(t_k - s)^{q-1}}{\Gamma(q)} \, | f_1(s, u_1(s), v_1(s)) - f_1(s, u_2(s), v_2(s)) | \, \mathrm{d}s +\\
&\sum_{0<t_k<t} (t - t_k) \, \Big| \int_{t_{k-1}}^{t_k} \frac{(t_k - s)^{q-2}}{\Gamma(q-1)} \, | f_1(s, u_1(s), v_1(s)) - f_1(s, u_2(s), v_2(s)) | \, \mathrm{d}s +\\
&\int_{t_k}^{t} \frac{(t_k - s)^{q-1}}{\Gamma(q)} \, | f_1(s, u_1(s), v_1(s)) - f_1(s, u_2(s), v_2(s)) | \, \mathrm{d}s +\\
&\sum_{0<t_k<t} | I_{1,k}(u_1(t_k^-)) - I_{1,k}(u_2(t_k^-)) | + \sum_{0<t_k<t} | (t - t_k) | \, | J_{1,k}(u_1(t_k^-)) - J_{1,k}(u_2(t_k^-)) | \leqslant\\
&\left[L_1 \Big[\frac{2(1+n)}{\Gamma(q+1)} + \frac{1+3n}{\Gamma(q)} \Big] + n(2L_2 + 3L_3) \right] \| (u_1, v_1) - (u_2, v_2) \|_{PC^1}。
\end{aligned}
$$

类似可得：

$$
\| Y(u_1, v_1) - Y(u_2, v_2) \|_{PC} \leqslant \left[l_1 \Big[\frac{2(1+n)}{\Gamma(p+1)} + \frac{1+3n}{\Gamma(p)} \Big] + n(2l_2 + 3l_3) \right] \| (u_1, v_1) - (u_2, v_2) \|_{PC^1},
$$

$$
\| X'(u_1, v_1) - X'(u_2, v_2) \|_{PC} \leqslant \left[L_1 \Big(\frac{1+n}{\Gamma(q+1)} + \frac{2+3n}{\Gamma(q)} \Big) + n(L_2 + 3L_3) \right] \| (u_1, v_1) - (u_2, v_2) \|_{PC^1},
$$

$$
\| Y'(u_1, v_1) - Y'(u_2, v_2) \|_{PC} \leqslant \left[l_1 \Big(\frac{1+n}{\Gamma(p+1)} + \frac{2+3n}{\Gamma(p)} \Big) + n(l_2 + 3l_3) \right] \| (u_1, v_1) - (u_2, v_2) \|_{PC^1},
$$

则有：

$$
\begin{aligned}
\| A(u_1, v_1) - A(u_2, v_2) \|_{PC^1} &= \max\{ \| X(u_1, v_1) - X(u_2, v_2) \|_{PC^1}, \| Y(u_1, v_1) - Y(u_2, v_2) \|_{PC^1} \} =\\
&\max\{ \max\{ \| X(u_1, v_1) - X(u_2, v_2) \|_{PC}, \| X'(u_1, v_1) - X'(u_2, v_2) \|_{PC} \},\\
&\quad \max\{ \| Y(u_1, v_1) - Y(u_2, v_2) \|_{PC}, \| Y'(u_1, v_1) - Y'(u_2, v_2) \|_{PC} \} \} \leqslant\\
&\quad\quad | \max\{ \max\{ \Lambda_1, \Lambda_3 \}, \max\{ \Lambda_2, \Lambda_4 \} \} | \; \| (u_1, v_1) - (u_2, v_2) \|_{PC^1} =\\
&\quad\quad | \max\{ \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4 \} | \; \| (u_1, v_1) - (u_2, v_2) \|_{PC^1}。
\end{aligned}
$$

其中：

$$
\Lambda_1 \leqslant L_1 \Big[\frac{2(1+n)}{\Gamma(q+1)} + \frac{1+3n}{\Gamma(q)} \Big] + n(2L_2 + 3L_3), \quad \Lambda_2 \leqslant l_1 \Big[\frac{2(1+n)}{\Gamma(p+1)} + \frac{1+3n}{\Gamma(p)} \Big] + n(2l_2 + 3l_3),
$$

$$
\Lambda_3 \leqslant L_1 \Big(\frac{1+n}{\Gamma(q+1)} + \frac{2+3n}{\Gamma(q)} \Big] + n(2L_2 + 3L_3), \quad \Lambda_4 \leqslant l_1 \Big(\frac{1+n}{\Gamma(p+1)} + \frac{2+3n}{\Gamma(p)} \Big] + n(2l_2 + 3l_3)。
$$

由于 $\max\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\} < 1$，则表明 $A(u, v)(t)$ 是一个压缩映射，因此，根据压缩映像原理定理得证。

定理 4　如果条件 H_1）和条件 H_2）成立，且 $n(2L_2 + 3L_3) < \frac{1}{2}, n(2l_2 + 3l_3) < \frac{1}{2}$，对于 $\forall (t, u, v) \in [0, 1] \times R \times R$，有 $| f_i(t, u, v) | \leqslant \| \mu \|_L, i = 1, 2$，其中 $\mu \in L([0, 1], R^+, R^+)$。则边值问题(1) 在$[0, 1]$中

至少有 1 个解。

证明 令

$$r_5 \geqslant \|\mu\|_L\left[\frac{2(1+n)}{\Gamma(q+1)}+\frac{1+3n}{\Gamma(q)}\right]+n(2M_2+3M_3),\quad r_6 \geqslant \|\mu\|_L\left[\frac{2(1+n)}{\Gamma(p+1)}+\frac{1+3n}{\Gamma(p)}\right]+n(2m_2+3m_3),$$

$$r_7 \geqslant \|\mu\|_L\left[\frac{1+n}{\Gamma(q+1)}+\frac{2+3n}{\Gamma(q)}\right]+n(M_2+3M_3),\quad r_8 \geqslant \|\mu\|_L\left[\frac{1+n}{\Gamma(p+1)}+\frac{2+3n}{\Gamma(p)}\right]+n(m_2+3m_3)。$$

$B_r = \{(u,v) \in PC^1(J,R) \times PC^1(J,R):\|(u,v)\| \leqslant r\}$。($r = \max\{r_5,r_6,r_7,r_8\}$)。定义在 B_r 上的映射 Φ 和 Ψ 如下：

$\Phi(u,v)(t) = (\Phi_1(u,v)(t),\Phi_2(u,v)(t))$，其中

$$\Phi_1(u,v)(t) = (1-t)\Big[\int_{t_n}^1 \frac{(1-s)^{q-1}}{\Gamma(q)}f_1(s,u(s),v(s))\mathrm{d}s+\int_{t_n}^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}f_1(s,u(s),v(s))\mathrm{d}s+$$

$$\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)}f_1(s,u(s),v(s))\mathrm{d}s+\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}(2-t_k)\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f_1(s,u(s),v(s))\mathrm{d}s+\Big]+$$

$$\chi(t)\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)}f_1(s,u(s),v(s))\mathrm{d}s+\chi(t)\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f_1(s,u(s),v(s))\mathrm{d}s+$$

$$\int_{t_k}^{t} \frac{(t_k-s)^{q-1}}{\Gamma(q)}f_1(s,u(s),v(s))\mathrm{d}s。$$

$$\Phi_2(u,v)(t) = (1-t)\Big[\int_{t_n}^1 \frac{(1-s)^{p-1}}{\Gamma(p)}f_2(s,u(s),v(s))\mathrm{d}s+\int_{t_n}^1 \frac{(1-s)^{p-2}}{\Gamma(p-1)}f_2(s,u(s),v(s))\mathrm{d}s+$$

$$\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{p-1}}{\Gamma(p)}f_2(s,u(s),v(s))\mathrm{d}s+\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{p-2}}{\Gamma(p-1)}f_2(s,u(s),v(s))\mathrm{d}s+\Big]+$$

$$\chi(t)\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{p-1}}{\Gamma(p)}f_2(s,u(s),v(s))\mathrm{d}s+\chi(t)\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{p-2}}{\Gamma(p-1)}f_2(s,u(s),v(s))\mathrm{d}s+$$

$$\int_{t_k}^{t} \frac{(t_k-s)^{p-1}}{\Gamma(p)}f_2(s,u(s),v(s))\mathrm{d}s。$$

$\Phi'(u,v)(t) = (\Phi'_1(u,v)(t),\Phi'_2(u,v)(t))$，其中

$$\Phi'_1(u,v)(t) = \int_{t_k}^{t} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f_1(s,u(s),v(s))\mathrm{d}s+\chi(t)\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f_1(s,u(s),v(s))\mathrm{d}s-$$

$$\Big[\int_{t_n}^1 \frac{(1-s)^{q-1}}{\Gamma(q)}f_1(s,u(s),v(s))\mathrm{d}s+\int_{t_n}^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}f_1(s,u(s),v(s))\mathrm{d}s+$$

$$\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)}f_1(s,u(s),v(s))\mathrm{d}s+\sum_{0<t_k<1}(2-t_k)\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f_1(s,u(s),v(s))\mathrm{d}s\Big],$$

$$\Phi'_2(u,v)(t) = \int_{t_k}^{t} \frac{(t_k-s)^{p-2}}{\Gamma(p-1)}f_2(s,u(s),v(s))\mathrm{d}s+\chi(t)\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{p-2}}{\Gamma(p-1)}f_2(s,u(s),v(s))\mathrm{d}s-$$

$$\Big[\int_{t_n}^1 \frac{(1-s)^{p-1}}{\Gamma(p)}f_2(s,u(s),v(s))\mathrm{d}s+\int_{t_n}^1 \frac{(1-s)^{p-2}}{\Gamma(p-1)}f_2(s,u(s),v(s))\mathrm{d}s+$$

$$\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{p-1}}{\Gamma(p)}f_2(s,u(s),v(s))\mathrm{d}s+\sum_{0<t_k<1}(2-t_k)\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{p-2}}{\Gamma(p-1)}f_2(s,u(s),v(s))\mathrm{d}s\Big]。$$

$\Psi(u,v)(t) = (\Psi_1(u,v)(t),\Psi_2(u,v)(t))$，其中

$$\Psi_1(u,v)(t) = (1-t)\Big(\sum_{0<t_k<1}I_{1,k}(u(t_k^-))+\sum_{0<t_k<1}(2-t_k)J_{1,k}(u(t_k^-))\Big)+\chi(t)\sum_{0<t_k<t}I_{1,k}(u(t_k^-))+$$

$$\chi(t)\sum_{0<t_k<t}(t-t_k)J_{1,k}(u(t_k^-))。$$

$$\Psi_2(u,v)(t) = (1-t)\Big(\sum_{0<t_k<1}I_{2,k}(v(t_k^-))+\sum_{0<t_k<1}(2-t_k)J_{2,k}(v(t_k^-))\Big)+\chi(t)\sum_{0<t_k<t}I_{2,k}(v(t_k^-))+$$

$$\chi(t)\sum_{0<t_k<t}(t-t_k)J_{2,k}(v(t_k^-))。$$

$\Psi'(u,v)(t) = (\Psi'_1(u,v)(t),\Psi'_2(u,v)(t))$，其中

$$\Psi'_1(u,v)(t) = \chi(t)\sum_{0<t_k<t}J_{1,k}(u(t_k^-))-\Big[\sum_{0<t_k<1}I_{1,k}(u(t_k^-))+\sum_{0<t_k<1}(2-t_k)J_{1,k}(u(t_k^-))\Big],$$

$$\Psi'_2(u,v)(t)=\chi(t)\sum_{0<t_k<t}J_{2,k}(v(t_k^-))-\Big[\sum_{0<t_k<1}I_{2,k}(v(t_k^-))+\sum_{0<t_k<1}(2-t_k)J_{2,k}(v(t_k^-))\Big]。$$

对于$(u_1,v_1),(u_2,v_2)\in B_r$，有

$$\|\Phi_1(u_1,v_1)+\Psi_1(u_2,v_2)\|_{PC}\leqslant\|\mu\|_L[(\frac{2(1+n)}{\Gamma(q+1)}+\frac{1+3n}{\Gamma(q)})]+n(2M_2+3M_3)\leqslant r_5\leqslant r,$$

$$\|\Phi_2(u_1,v_1)+\Psi_2(u_2,v_2)\|_{PC}\leqslant\|\mu\|_L[(\frac{2(1+n)}{\Gamma(p+1)}+\frac{1+3n}{\Gamma(p)})]+n(2m_2+3m_3)\leqslant r_6\leqslant r,$$

$$\|\Phi'_1(u_1,v_1)+\Psi'_1(u_2,v_2)\|_{PC}\leqslant\|\mu\|_L[(\frac{1+n}{\Gamma(q+1)}+\frac{2+3n}{\Gamma(q)})]+n(M_2+3M_3)\leqslant r_7\leqslant r,$$

$$\|\Phi'_2(u_1,v_1)+\Psi'_2(u_2,v_2)\|_{PC}\leqslant\|\mu\|_L[(\frac{1+n}{\Gamma(p+1)}+\frac{2+3n}{\Gamma(p)})]+n(m_2+3m_3)\leqslant r_8\leqslant r。$$

所以

$$\begin{aligned}&\|\Phi(u_1,v_1)+\Psi(u_2,v_2)\|_{PC^1}=\max\{\|\Phi_1(u_1,v_1)+\Psi_1(u_2,v_2)\|_{PC^1},\|\Phi_2(u_1,v_1)+\Psi_2(u_2,v_2)\|_{PC^1}\}=\\&\qquad\max\{\max\{\|\Phi_1(u_1,v_1)+\Psi_1(u_2,v_2)\|_{PC},\|\Phi'_1(u_1,v_1)+\Psi'_1(u_2,v_2)\|_{PC}\},\\&\qquad\quad\max\{\|\Phi_2(u_1,v_1)+\Psi_2(u_2,v_2)\|_{PC},\|\Phi'_2(u_1,v_1)+\Psi'_2(u_2,v_2)\|_{PC}\}\}\leqslant r,\end{aligned}$$

即 $\Phi(u_1,v_1)(t)+\Psi(u_2,v_2)(t)\in B_r$。

因为 $\|\Psi(u_1,v_1)-\Psi(u_2,v_2)\|_{PC^1}=\max\{\|\Psi_1(u_1,v_1)+\Psi_1(u_2,v_2)\|_{PC^1},\|\Psi_2(u_1,v_1)+\Psi_2(u_2,v_2)\|_{PC^1}\}=$

$$\begin{aligned}&\max\{\max\{\|\Psi_1(u_1,v_1)+\Psi_2(u_2,v_2)\|_{PC},\|\Psi_1(u_1,v_1)-\Psi_1(u_2,v_2)\|_{PC}\},\\&\quad\max\{\|\Psi_2(u_1,v_1)+\Psi_2(u_2,v_2)\|_{PC},\|\Psi_2(u_1,v_1)-\Psi_2(u_2,v_2)\|_{PC}\}\}\leqslant\\&\quad|\max\{\max\{n(2L_2+3L_3),n(L_2+3L_3)\},\\&\qquad\max\{n(2l_2+3l_3),n(l_2+3l_3)\}\}|\ \|(u_1,v_1)-(u_2,v_2)\|_{PC^1}\leqslant\\&|\max\{\max\{n(2L_2+3L_3),n(L_2+3L_3)\},\max\{n(2l_2+3l_3),n(l_2+3l_3)\}\}|<1,\end{aligned}$$

所以映射 Ψ 是压缩映射。

因为 f_i 是连续有界函数，所以映射 Φ 是连续的，并且在 B_r 上有界。

$$\begin{aligned}&\|\Phi(u,v)\|_{PC^1}=\max\{\|\Phi_1(u,v)\|_{PC^1},\|\Phi_2(u,v)\|_{PC^1}\}=\\&\quad\max\{\max\{\|\Phi_1(u,v)\|_{PC},\|\Phi'_1(u,v)\|_{PC}\},\max\{\|\Phi_2(u,v)\|_{PC},\|\Phi'_2(u,v)\|_{PC}\}\}\leqslant\\&\quad\max\{\max\{\|\mu\|_L[\frac{2(1+n)}{\Gamma(q+1)}+\frac{1+3n}{\Gamma(q)}],\|\mu\|_L[\frac{1+n}{\Gamma(q+1)}+\frac{2+3n}{\Gamma(q)})\},\\&\qquad\max\{\|\mu\|_L[(\frac{2(1+n)}{\Gamma(p+1)}+\frac{1+3n}{\Gamma(p)})],\|\mu\|_L(\frac{1+n}{\Gamma(p+1)}+\frac{2+3n}{\Gamma(p)})\}\}。\end{aligned}$$

证明映射 Φ 的紧性。

令 $\Omega=[0,1]\times B_r$，定义 $\sup_{(t,u,v)\in\Omega}|f_1(t,u,v)|=f_{1\max}$，$\sup_{(t,u,v)\in\Omega}|f_2(t,u,v)|=f_{2\max}$。

取 $\tau_1,\tau_2\in[0,1]$，$\tau_1<\tau_2$，则有：

$$\begin{aligned}&\|\Phi_1(u,v)(\tau_2)-\Phi_1(u,v)(\tau_1)\|_{PC}=\\&\quad\Big|(\tau_1-\tau_2)\Big[\int_{t_n}^1\frac{(1-s)^{q-1}}{\Gamma(q)}f_1(s,u(s),v(s))\mathrm{d}s+\int_{t_n}^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}f_1(s,u(s),v(s))\mathrm{d}s+\\&\quad\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}f_1(s,u(s),v(s))\mathrm{d}s+\sum_{0<t_k<1}\int_{t_{k-1}}^{t_k}(2-t_k)\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f_1(s,u(s),v(s))\mathrm{d}s\Big]+\\&\quad\frac{1}{\Gamma(q)}\chi(t)\sum_{\tau_1<t_k<\tau_2}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}f_1(s,u(s),v(s))\mathrm{d}s+\\&\quad\frac{1}{\Gamma(q-1)}\chi(t)\sum_{0<t_k<\tau_1}(\tau_2-\tau_1)\int_{t_{k-1}}^{t_k}(t_k-s)^{q-2}f_1(s,u(s),v(s))\mathrm{d}s+\\&\quad\frac{1}{\Gamma(q-1)}\chi(t)\sum_{\tau_1<t_k<\tau_2}(\tau_2-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{q-2}f_1(s,u(s),v(s))\mathrm{d}s+\\&\quad\frac{1}{\Gamma(q)}\Big[\int_{\tau_1}^{\tau_2}(\tau_2-s)^{q-1}f_1(s,u(s),v(s))\mathrm{d}s+\int_{t_k}^{\tau_1}[(\tau_2-s)^{q-1}-(\tau_1-s)^{q-1}]f_1(s,u(s),v(s))\mathrm{d}s\Big]\Big|\leqslant\\&f_{1\max}\Big\{(\frac{1+n}{\Gamma(q+1)}+\frac{1+2n}{\Gamma(q)})|\tau_1-\tau_2|+\frac{1}{\Gamma(q+1)}\sum_{\tau_1<t_k<\tau_2}|(t_k-t_{k-1})^q|+\frac{1}{\Gamma(q)}\sum_{0<t_k<\tau_1}|(\tau_2-\tau_1)(t_k-t_{k-1})^{q-1}|+\\&\frac{1}{\Gamma(q)}\sum_{\tau_1<t_k<\tau_2}|(\tau_2-t_k)(t_k-t_{k-1})^{q-1}|+\frac{1}{\Gamma(q+1)}[(\tau_2-\tau_1)^q+|-(\tau_2-\tau_1)^q+(\tau_2-t_k)^q-(\tau_1-t_k)^q|]\Big\}。\end{aligned}$$

类似可得，

$$\| \Phi_2(u,v)(\tau_2) - \Phi_2(u,v)(\tau_1) \|_{PC} \leqslant$$

$$f_{2\max}\left\{ \left(\frac{1+n}{\Gamma(p+1)} + \frac{1+2n}{\Gamma(p)}\right) | \tau_1 - \tau_2 | + \frac{1}{\Gamma(p+1)} \sum_{\tau_1 < t_k < \tau_2} | (t_k - t_{k-1})^p | + \frac{1}{\Gamma(p)} \sum_{0 < t_k < \tau_1} | (\tau_2 - \tau_1)(t_k - t_{k-1})^{p-1} | + \frac{1}{\Gamma(p)} \sum_{\tau_1 < t_k < \tau_2} | (\tau_2 - t_k)(t_k - t_{k-1})^{p-1} | + \frac{1}{\Gamma(p+1)} [(\tau_2 - \tau_1)^q + | - (\tau_2 - \tau_1)^p + (\tau_2 - t_k)^p - (\tau_1 - t_k)^p |] \right\}。$$

$$\| \Phi'_1(u,v)(\tau_2) - \Phi'_1(u,v)(\tau_1) \|_{PC} \leqslant$$

$$f_{1\max}\left\{ \frac{1}{\Gamma(q)} \sum_{\tau_1 < t_k < \tau_2} | (t_k - t_{k-1})^{q-1} | + \frac{1}{\Gamma(q)} [(\tau_2 - \tau_1)^{q-1} + | - (\tau_2 - \tau_1)^{q-1} + (\tau_2 - t_k)^{q-1} - (\tau_1 - t_k)^{q-1} |] \right\},$$

$$\| \Phi'_2(u,v)(\tau_2) - \Phi'_2(u,v)(\tau_1) \|_{PC} \leqslant$$

$$f_{2\max}\left\{ \frac{1}{\Gamma(p)} \sum_{\tau_1 < t_k < \tau_2} | (t_k - t_{k-1})^{p-1} | + \frac{1}{\Gamma(p)} [(\tau_2 - \tau_1)^{p-1} + | - (\tau_2 - \tau_1)^{p-1} + (\tau_2 - t_k)^{p-1} - (\tau_1 - t_k)^{p-1} |] \right\}。$$

则 $\| \Phi(u,v)(\tau_2) - \Phi(u,v)(\tau_1) \|_{PC^1} =$

$$\max\{ \| \Phi_1(u,v)(\tau_2) - \Phi_1(u,v)(\tau_1) \|_{PC^1}, \| \Phi_2(u,v)(\tau_2) - \Phi_2(u,v)(\tau_1) \|_{PC^1} \} =$$

$$\max\{\max\{ \| \Phi_1(u,v)(\tau_2) - \Phi_1(u,v)(\tau_1) \|_{PC}, \| \Phi'_1(u,v)(\tau_2) - \Phi'_1(u,v)(\tau_1) \|_{PC} \},$$

$$\max\{ \| \Phi_2(u,v)(\tau_2) - \Phi_2(u,v)(\tau_1) \|_{PC}, \| \Phi'_2(u,v)(\tau_2) - \Phi'_2(u,v)(\tau_1) \|_{PC} \}\}。$$

与(u,v)无关，则Φ在B_r上是相对紧的。因此，由 Arzela-Ascoil 定理可知，Φ在B_r上是紧的，则Φ是全连续的。所以满足定理 3 的所有条件，则边值问题(1) 在$[0,1]$中至少有 1 个解。

4 举 例

例 1 考虑分数阶脉冲微分方程组边值问题

$$\begin{cases} {}^cD^{\frac{3}{2}}u(t) = \dfrac{1}{(t+6)^2}\dfrac{| v(t) |}{1+| v(t) |}, & t \in J', t \neq \dfrac{1}{3}, \\ {}^cD^{\frac{7}{4}}v(t) = \dfrac{1}{(t+8)^2}\dfrac{| u(t) |}{1+| u(t) |}, & t \in J', t \neq \dfrac{1}{3}, \\ \Delta u(\dfrac{1}{3}) = \dfrac{| u(t) |}{15+| u(t) |}, & \Delta v(\dfrac{1}{3}) = \dfrac{| v(t) |}{16+| v(t) |}, \\ \Delta u'(\dfrac{1}{3}) = \dfrac{| u(t) |}{17+| u(t) |}, & \Delta v'(\dfrac{1}{3}) = \dfrac{| v(t) |}{18+| v(t) |}, \\ u(0) + u'(0) = 0, & v(0) = v'(0) = 0, \\ u(1) + u'(1) = 0, & v(1) = v'(1) = 0 \end{cases}$$

解：$L_1 = \frac{1}{36}, L_2 = \frac{1}{15}, L_3 = \frac{1}{17}, l_1 = \frac{1}{64}, l_2 = \frac{1}{16}, l_3 = \frac{1}{18}, q = \frac{3}{2}, p = \frac{7}{4}, n = 1$，则：

$$\Lambda_1 = L_1\left(\frac{2(1+n)}{\Gamma(q+1)} + \frac{1+3n}{\Gamma(q)}\right) + n(2L_2 + 3L_3) = \left(\frac{10}{27\sqrt{\pi}} + \frac{79}{255}\right) < 1,$$

$$\Lambda_2 = l_1\left(\frac{2(1+n)}{\Gamma(p+1)} + \frac{1+3n}{\Gamma(p)}\right) + n(2l_2 + 3l_3) = \left(\frac{19}{84\Gamma(\frac{3}{4})} + \frac{7}{24}\right) < 1,$$

$$\Lambda_3 = L_1\left(\frac{1+n}{\Gamma(q+1)} + \frac{2+3n}{\Gamma(q)}\right) + n(L_2 + 3L_3) = \left(\frac{19}{54\sqrt{\pi}} + \frac{62}{255}\right) < 1,$$

$$\Lambda_4 = l_1\left(\frac{1+n}{\Gamma(p+1)} + \frac{2+3n}{\Gamma(p)}\right) + n(l_2 + 3l_3) = \left(\frac{43}{336\Gamma(\frac{3}{4})} + \frac{11}{48}\right) < 1。$$

满足定理 4 的条件，此边值问题有唯一解。

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