

分数阶差分方程解的振动性

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摘要:分数阶微积分是研究任意阶微分和积分性质及应用的一种理论,它可以更加精确的描述一些系统的物理特性,更加适应系统的变化,可以应用于描述生物医学中的肿瘤生长(生长刺激与生长抑制)过程。为了研究 2 类分数阶差分方程解的振动性,主要利用反证法,即假设方程有非振动解,对于第 1 类方程首先确定函数符号,通过构造 Riccati 函数,对其求差分,利用函数满足的条件得到矛盾,即假设不成立,验证了解的振动性。对于第 2 类带有初值条件的方程,首先证明了与该分数阶差分方程等价的和分形式,然后分别考虑 $0 < \alpha \leq 1$ 和 $\alpha > 1$ 两种情况,运用 Stirling 公式及阶乘函数的性质,放大处理得到与已知条件相矛盾,假设不成立,获得分数阶差分方程有界解振动的充分条件。以上结果优化了相关结论,丰富了相关成果,并把结果应用到具体方程之中,验证了方程解的振动性质。

关键词:定性理论;分数阶;振动性;差分;微积分

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Oscillation results for certain fractional difference equations

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Abstract: Fractional calculus is a theory that studies the properties and application of arbitrary order differentiation and integration. It can describe the physical properties of some systems more accurately, and better adapt to changes in the system, playing an important role in many fields. For example, it can describe the process of tumor growth (growth stimulation and growth inhibition) in biomedical science. The oscillation of solutions of two kinds of fractional difference equations is studied, mainly using the proof by contradiction, that is, assuming the equation has a nonstationary solution. For the first kind of equation, the function symbol is firstly determined, and by constructing the Riccati function, the difference is calculated. Then the condition of the function is used to satisfy the contradiction, that is, the assumption is false, which verifies the oscillation of the solution. For the second kind of equation with initial condition, the equivalent fractional sum form of the fractional difference equation are firstly proved. With considering $0 < \alpha \leq 1$ and $\alpha > 1$, respectively, by using the properties of Stirling formula and factorial function, the contradictory is got through enhanced processing, namely the assuming is not established, and the sufficient condition for the bounded solutions of the fractional difference equation is obtained. The above results will optimize the relevant conclusions and enrich the relevant results. The results are applied to the specific equations, and the oscillation of the solutions of equations is proved.

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分数阶微积分可应用于越来越多的领域中,例如:生物学、物理学、粘弹性、控制理论等方面^[1-5]。许多学者研究了分数阶微分方程的各种性质,例如:解的存在唯一性、解的稳定性、解对初值的连续依赖性、解的振动性等^[6-13]。当物质拥有记忆和遗传效应或者诸如质量扩散与热传导的过程,在用整数阶微分方程描述时不能精确的表征其中的物理特性,这就需要对传统的整数阶微积分进行推广,以便更好地描述这些现象。

虽然分数阶差分方程解的振动性在工程领域中有着重要的作用,但是,关于分数阶差分方程解的振动性的相关理论较少。

2012年, MARIAN 等^[14]研究了形式如下的非线性分数阶差分方程解的振动性,

$$\begin{cases} \Delta^\alpha x(t) + f_1(t, x(t+\alpha)) = v(t) + f_1(t, x(t+\alpha)), \\ \Delta^{\alpha-1} x(t) |_{t=0} = x_0. \end{cases}$$

2014年, SAGAYARAJ 等^[15]讨论了形式如下的分数阶差分方程解的振动性,

$$\Delta(p(t)g(\Delta^\alpha x(t)) + q(t)f(\sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s))) = 0.$$

2016年, LI^[16]考虑了带有强迫项的分数阶差分方程的解的振动性,

$$\begin{cases} (1+p(t))\Delta(\Delta^\alpha x(t)) + p(t)\Delta^\alpha x(t) + f(t, x(t)) = g(t), \\ \Delta^{\alpha-1} x(t) |_{t=0} = x_0, \quad 0 < \alpha < 1. \end{cases}$$

同年, SELVAM 等^[17]研究了形式如下的分数阶差分方程解的振动性,

$$\Delta(a(t)(\Delta^\alpha x(t))^r) + p(t)(\Delta^\alpha x(t))^r + q(t)f(\sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s)) = 0.$$

受以上学者的启发,笔者讨论 2 类分数阶差分方程解的振动性,得到了其振动的充分条件。

首先考虑了如下形式的方程:

$$\Delta[r(t)(\Delta^\alpha x(t))^\eta] + q(t)f(z(t)) = 0, \quad t \in N_0, \quad (1)$$

其中 Δ^α 是 Riemann-Liouville 定义的 α 阶差分算子, $0 < \alpha < 1, \eta > 0$ 是正奇整数的商, $N_0 = \{0, 1, 2, \dots\}$ 。

对于方程(1)假设有以下条件,

H) $r(t), q(t)$ 是正函数, $z(t) = \Gamma(1-\alpha)\Delta^{\alpha-1}x(t)$ 且 $f: \mathbf{R} \rightarrow \mathbf{R}$ 是连续函数, 对于 $z \neq 0$ 满足 $f(z(t))/z(t+1) \geq K$, 其中 $K > 0$ 。

其次, 笔者讨论了带有初值的分数阶差分方程, 形式如式(2)所示:

$$\begin{cases} \Delta[r(t)\Delta^\alpha x(t)] + q(t)f(z(t)) = v(t), \quad t \in N_0, \\ \Delta^{\alpha-k} x(t) |_{t=0} = b_k, \quad k = 1, 2, \dots, m, \\ \Delta^\alpha x(t) |_{t=0} = b_0, \end{cases} \quad (2)$$

其中 $m-1 < \alpha \leq m, m \geq 1$ 是整数, v 是实函数, 并且满足以下条件:

H') $r(t), q(t)$ 是正函数, $z(t) = \sum_{s=0}^{t-1} (t-s-1)^{-\alpha} x(s)$ 且 $f: \mathbf{R} \rightarrow \mathbf{R}$ 是连续函数, 对 $z \neq 0$ 满足 $0 < |f(z(t))/z(t)| \leq K$, 其中 $K > 0$ 。

如果解 $x(t)$ 既不最终正的, 也不最终负的, 则称 $x(t)$ 是方程(1)(或者方程(2))的振动解; 否则, 称其为非振动解。

1 预备知识

定义 1^[18] 设 $v > 0$, 函数的 v 阶和分定义为

$$\Delta^{-v} f(t) = \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t-s-1)^{(v-1)} f(s), \quad (3)$$

其中 $f(t)$ 和 $\Delta^{-v} f(t)$ 分别定义在 $s = a \pmod{1}$ 和 $t = a + v \pmod{1}$ 上, $t^{(v)} = \frac{\Gamma(t+1)}{\Gamma(t-v+1)}$ 。

定义 2^[18] 设 $\mu > 0, m-1 < \mu \leq m$, 其中 m 是正整数, $m = [\mu]$. 设 $v = m - \mu, f(t)$ 的 μ 阶差分定义为

$$\Delta^\mu f(t) = \Delta^{m-v} f(t) = \Delta^m \Delta^{-v} f(t). \quad (4)$$

引理 1^[18] 对于任意实数 $v > 0$, 任意正整数 p , 以下等式成立:

$$\Delta^{-v} \Delta^p f(t) = \Delta^p \Delta^{-v} f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{(v-p+k)}}{\Gamma(v+k-p+1)} \Delta^k f(a), \quad (5)$$

其中 $f(t)$ 定义在 N_a 上。

引理 2 方程(2)的等价和分形式为

$$x(t) = \sum_{k=0}^{m-1} \frac{b_{m-k} t^{(\alpha-m+k)}}{\Gamma(\alpha-m+k+1)} + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-a} (t-s-1)^{(\alpha-1)} \left[\frac{r(0)}{r(s)} b_0 + \frac{1}{r(s)} \sum_{\xi=0}^{s-1} (v(\xi) - f(z(\xi))q(\xi)) \right]. \quad (6)$$

证明 将 Δ^{-1} 算子应用到方程(2)两边, 得到:

$$\Delta^{-1} \Delta [r(t) \Delta^\alpha x(t)] = \Delta^{-1} [-q(t)f(z(t)) + v(t)], \quad (7)$$

由定义 1 和引理 1, 得到:

$$\Delta^\alpha x(t) = \frac{r(0)}{r(t)} b_0 + \frac{1}{r(t)} \sum_{s=0}^{t-1} (-q(s)f(z(s)) + v(s)), \quad (8)$$

应用 $\Delta^{-\alpha}$ 到式(8)两边, 有

$$\Delta^{-\alpha} \Delta^\alpha x(t) = \Delta^{-\alpha} \left[\frac{r(0)}{r(t)} b_0 + \frac{1}{r(t)} \sum_{s=0}^{t-1} (-q(s)f(z(s)) + v(s)) \right], \quad (9)$$

由引理 1, 得到:

$$\begin{aligned} \Delta^{-\alpha} \Delta^\alpha x(t) &= \Delta^{-\alpha} \Delta^m \Delta^{-(m-\alpha)} x(t) = \\ &= \Delta^m \Delta^{-\alpha} \Delta^{-(m-\alpha)} x(t) - \sum_{k=0}^{m-1} \frac{t^{(\alpha-m+k)}}{\Gamma(\alpha-m+k+1)} \Delta^k \Delta^{-(m-\alpha)} x(0) = \\ &= x(t) - \sum_{k=0}^{m-1} \frac{t^{(\alpha-m+k)}}{\Gamma(\alpha-m+k+1)} b_{m-k}, \end{aligned} \quad (10)$$

应用定义 1, 有:

$$\begin{aligned} \Delta^{-\alpha} \left[\frac{r(0)}{r(t)} b_0 + \frac{1}{r(t)} \sum_{s=0}^{t-1} (-q(s)f(z(s)) + v(s)) \right] &= \\ \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-a} (t-s-1)^{(\alpha-1)} \left[\frac{r(0)}{r(s)} b_0 + \frac{1}{r(s)} \sum_{\xi=0}^{s-1} (-q(\xi)f(z(\xi)) + v(\xi)) \right], \end{aligned} \quad (11)$$

由式(10)和式(11), 得到等式(6)。

引理 3^[19] 对于 $\alpha, \beta, t > 0$ 有:

$$t^{(-\beta)} > (t+\alpha)^{(-\beta)}. \quad (12)$$

引理 4^[20] 对于分数阶函数, 有以下性质:

$$t^{(\beta+\gamma)} = (t-\gamma)^{(\beta)} t^{(\gamma)}. \quad (13)$$

2 主要结论

定理 1 设条件 H) 及以下条件满足:

$$\sum_{s=0}^{\infty} r^{-\frac{1}{\gamma}}(s) = \infty, \quad (14)$$

$$\sum_{s=0}^{\infty} q(s) = \infty, \quad (15)$$

那么方程(1)的每个解都是振动的。

证明 假设 $x(t)$ 是方程(1)的 1 个非振动解。首先, 假设 $x(t)$ 是最终正解, 则存在 $t_1 > t_0$ 使得 $x(t) > 0$, 所以对 $t \in [t_1, \infty)$ 有 $z(t) > 0$, 其中 z 是条件 H) 中定义的函数。对 $t \in [t_1, \infty)$ 有:

$$\Delta[r(t)(\Delta^\alpha x(t))^\eta] = -q(t)f(z(t)) < 0, \quad (16)$$

因此 $r(t)(\Delta^\alpha x(t))^\eta$ 在 $[t_1, \infty)$ 上是严格递减函数并且最终定号。由于 $t \in [t_1, \infty)$ 时 $r(t) > 0, \eta$ 是正奇整数的商, 则 $\Delta^\alpha x(t)$ 最终定号。

下面证明:

$$\Delta^\alpha x(t) > 0. \quad (17)$$

如果 $\Delta^\alpha x(t) < 0, t \in [t_1, \infty)$, 那么 $\Delta^\alpha x(t)$ 是最终负的, 存在 $t_2 \in [t_1, \infty)$ 使得 $\Delta^\alpha x(t_2) < 0$ 。

因为 $r(t)(\Delta^\alpha x(t))^\eta$ 在 $[t_1, \infty)$ 上是严格递减的, 则有:

$$r(t)(\Delta^\alpha x(t))^\eta \leq r(t_2)(\Delta^\alpha x(t_2))^\eta = c < 0, \quad t \in [t_2, \infty).$$

由条件 H) 得到: $\Delta z(t) = \Gamma(1-\alpha)\Delta^\alpha x(t) \leq \Gamma(1-\alpha)\left(\frac{c}{r(t)}\right)^{\frac{1}{\eta}}$, 对上式两边同时从 t_2 到 $(t-1)$ 求和, 得

$$z(t) \leq z(t_2) + \sum_{s=t_2}^{t-1} \Gamma(1-\alpha)\left(\frac{c}{r(s)}\right)^{\frac{1}{\eta}}, \text{ 令式(17)} t \rightarrow \infty,$$

$$\liminf_{t \rightarrow \infty} z(t) \leq \liminf_{t \rightarrow \infty} \left[z(t_2) + \sum_{s=t_2}^{t-1} \Gamma(1-\alpha)\left(\frac{c}{r(s)}\right)^{\frac{1}{\eta}} \right] = -\infty,$$

与 $z(t) > 0$ 矛盾, 所以式(17) 成立。因此 $\Delta z(t) = \Gamma(1-\alpha)\Delta^\alpha x(t) > 0$, 即 $z(t)$ 是增函数。

定义 Riccati 函数:

$$w(t) = \frac{r(t)(\Delta^\alpha x(t))^\eta}{z(t)}, \quad t \in [t_1, \infty), \quad (18)$$

则 $w(t) > 0, t \in [t_1, \infty)$ 。由式(16)、式(18) 和条件 H), 得到:

$$\begin{aligned} \Delta w(t) &= \frac{\Delta[r(t)(\Delta^\alpha x(t))^\eta]z(t) - r(t)(\Delta^\alpha x(t))^\eta \Delta z(t)}{z(t)z(t+1)} = \\ &= \frac{\Delta[r(t)(\Delta^\alpha x(t))^\eta]}{z(t+1)} - \frac{r(t)(\Delta^\alpha x(t))^\eta}{z(t)} + \frac{r(t)(\Delta^\alpha x(t))^\eta}{z(t+1)} = \\ &= -\frac{q(t)f(z(t))}{z(t+1)} - w(t) + \frac{z(t)}{z(t+1)}w(t), \end{aligned}$$

故有:

$$w(t+1) = -\frac{q(t)f(z(t))}{z(t+1)} + \frac{z(t)}{z(t+1)}w(t) \leq w(t) - Kq(t),$$

即

$$w(t+1) - w(t) \leq -Kq(t), \quad (19)$$

令上式 $t = s$, 两边同时对 s 从 t_2 到 $(t-1)$ 求和, 得到: $w(t) \leq w(t_2) - K \sum_{s=t_2}^{t-1} q(s)$, 那么 $\liminf_{t \rightarrow \infty} w(t) \leq$

$$\liminf_{t \rightarrow \infty} [w(t_2) - K \sum_{s=t_2}^{t-1} q(s)] = -\infty, \text{ 与 } w(t) > 0 \text{ 矛盾。}$$

假设 $x(t)$ 是方程(1) 的一个最终负解, 则存在 $t_1 > t_0$, 使得 $x(t) < 0$, 所以对 $t \in [t_1, \infty)$ 有 $z(t) < 0$, 其中 z 是条件 H) 中定义的函数, 则对 $t \in [t_1, \infty)$ 有:

$$\Delta[r(t)(\Delta^\alpha x(t))^\eta] = -q(t)f(z(t)) > 0, \quad (20)$$

因此 $r(t)(\Delta^\alpha x(t))^\eta$ 在 $[t_1, \infty)$ 上是严格递增函数并且最终定号。由于 $t \in [t_1, \infty)$ 时 $r(t) > 0, \eta$ 是正奇整数的商, 则 $\Delta^\alpha x(t)$ 最终定号。

下面证明

$$\Delta^\alpha x(t) < 0, \quad t \in [t_1, \infty). \quad (21)$$

如果 $\Delta^\alpha x(t) > 0, t \in [t_1, \infty)$, 那么 $\Delta^\alpha x(t)$ 是最终正的, 存在 $t_2 \in [t_1, \infty)$ 使得 $\Delta^\alpha x(t_2) > 0$ 。因为 $r(t)(\Delta^\alpha x(t))^\eta$ 在 $[t_1, \infty)$ 上是严格递增的, 则有 $r(t)(\Delta^\alpha x(t))^\eta \geq r(t_2)(\Delta^\alpha x(t_2))^\eta = c > 0, t \in [t_2, \infty)$ 。由条件 H) 得到: $\frac{\Delta z(t)}{\Gamma(1-\alpha)} = \Delta^\alpha x(t) \geq \left(\frac{c}{r(t)}\right)^{\frac{1}{\eta}}$, 所以有 $\Delta z(t) \geq \Gamma(1-\alpha)\left(\frac{c}{r(t)}\right)^{\frac{1}{\eta}}$ 。对上式两边同时从 t_2 到 $(t-1)$

求和, 得到:

$$z(t) \geq z(t_2) + \sum_{s=t_2}^{t-1} \Gamma(1-\alpha) \left(\frac{c}{r(s)}\right)^{\frac{1}{\eta}}.$$

令上式 $t \rightarrow \infty$,

$$\liminf_{t \rightarrow \infty} z(t) \geq \liminf_{t \rightarrow \infty} \left[z(t_2) + \sum_{s=t_2}^{t-1} \Gamma(1-\alpha) \left(\frac{c}{r(s)}\right)^{\frac{1}{\eta}} \right] = \infty,$$

与 $z(t) < 0$ 矛盾, 所以式(21) 成立. 因此 $\Delta z(t) = \Gamma(1-\alpha) \Delta^\alpha x(t) < 0$, 即 $z(t)$ 是减函数.

定义 Riccati 函数: $w(t) = \frac{r(t)(\Delta^\alpha x(t))^\eta}{z(t)}$, $t \in [t_1, \infty)$, 则 $w(t) > 0, t \in [t_1, \infty)$.

证明过程同以上证明类似, 在这里省略, 定理得证.

定理 2 假设条件 H') 满足, 对任意的 C_1, C_2 如果对于充分大的 T 有:

$$\liminf_{t \rightarrow \infty} \sum_{s=T}^{t-\alpha} \frac{t^{1-\alpha}(t-s-1)^{(\alpha-1)}}{r(s)} \left[C_1 + C_2 \sum_{\xi=0}^{s-1} q(\xi) \sum_{\eta=0}^{\xi-1} (\xi-\eta-1)^{(-\alpha)} + \sum_{\xi=0}^{s-1} v(\xi) \right] = -\infty, \tag{22}$$

$$\limsup_{t \rightarrow \infty} \sum_{s=T}^{t-\alpha} \frac{t^{1-\alpha}(t-s-1)^{(\alpha-1)}}{r(s)} \left[C_1 + C_2 \sum_{\xi=0}^{s-1} q(\xi) \sum_{\eta=0}^{\xi-1} (\xi-\eta-1)^{(-\alpha)} + \sum_{\xi=0}^{s-1} v(\xi) \right] = \infty, \tag{23}$$

则方程 (2) 的每个有界解都是振动的.

证明 设 $x(t)$ 是方程(2) 的有界非振动解. 则存在 M_1, M_2 使得:

$$M_1 \leq x(t) \leq M_2. \tag{24}$$

首先, 假设 $x(t)$ 是方程(2) 的最终正解, 则存在 t_1 使 $x(t) > 0, z(t) > 0, t \geq t_1$, 将式(6) 两边同时乘以 $t^{1-\alpha} \Gamma(\alpha)$ 得到:

$$\begin{aligned} t^{1-\alpha} \Gamma(\alpha) x(t) &= t^{1-\alpha} \Gamma(\alpha) \sum_{k=0}^{m-1} \frac{b_{m-k} t^{(\alpha-m+k)}}{\Gamma(\alpha-m+k+1)} + \\ & t^{1-\alpha} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[\frac{r(0)}{r(s)} b_0 + \frac{1}{r(s)} \sum_{\xi=0}^{s-1} (v(\xi) - f(z(\xi))q(\xi)) \right] = \\ & t^{1-\alpha} \Gamma(\alpha) \sum_{k=0}^{m-1} \frac{b_{m-k} t^{(\alpha-m+k)}}{\Gamma(\alpha-m+k+1)} + \\ & \sum_{s=0}^{t_1-1} t^{1-\alpha} (t-s-1)^{(\alpha-1)} \left[\frac{r(0)}{r(s)} b_0 + \frac{1}{r(s)} \sum_{\xi=0}^{s-1} (v(\xi) - f(z(\xi))q(\xi)) \right] + \\ & \sum_{s=t_1}^{t-\alpha} t^{1-\alpha} (t-s-1)^{(\alpha-1)} \left[\frac{r(0)}{r(s)} b_0 + \frac{1}{r(s)} \sum_{\xi=0}^{s-1} (v(\xi) - f(z(\xi))q(\xi)) \right] \leq \\ & \Phi(t) + \Psi(t, t_1) + \sum_{s=t_1}^{t-\alpha} t^{1-\alpha} (t-s-1)^{(\alpha-1)} \left[\frac{r(0)}{r(s)} b_0 + \right. \\ & \left. \frac{K}{r(s)} \sum_{\xi=0}^{s-1} q(\xi) \sum_{\eta=0}^{\xi-1} (\xi-\eta-1)^{(-\alpha)} x(\xi) + \frac{1}{r(s)} \sum_{\xi=0}^{s-1} v(\xi) \right], \tag{25} \end{aligned}$$

其中:

$$\Phi(t) = t^{1-\alpha} \Gamma(\alpha) \sum_{k=0}^{m-1} \frac{b_{m-k} t^{(\alpha-m+k)}}{\Gamma(\alpha-m+k+1)},$$

$$\Psi(t, t_1) = \sum_{s=0}^{t_1-1} t^{1-\alpha} (t-s-1)^{(\alpha-1)} \left[\frac{r(0)}{r(s)} b_0 + \frac{1}{r(s)} \sum_{\xi=0}^{s-1} (v(\xi) - f(z(\xi))q(\xi)) \right].$$

则有:

$$0 < t^{1-\alpha} \Gamma(\alpha) x(t) \leq \Phi(t) + \Psi(t, t_1) + \sum_{s=t_1}^{t-\alpha} t^{1-\alpha} (t-s-1)^{(\alpha-1)} \left[\frac{r(0)}{r(s)} b_0 + \frac{K}{r(s)} \sum_{\xi=0}^{s-1} q(\xi) \sum_{\eta=0}^{\xi-1} (\xi-\eta-1)^{(-\alpha)} x(\xi) + \frac{1}{r(s)} \sum_{\xi=0}^{s-1} v(\xi) \right]. \tag{26}$$

令 $t_2 > t_1$, 下面分别考虑 $0 < \alpha \leq 1$ 和 $\alpha > 1$ 的情况.

i) $0 < \alpha \leq 1$, 则 $m = 1, \Phi(t) = t^{1-\alpha} t^{(\alpha-1)} b_1$, 由文献[16] 可知, $\lim_{t \rightarrow \infty} t^{1-\alpha} t^{(\alpha-1)} = 1$, 即存在 $M > 0$ 使

$$(t-s)^{1-\alpha}(t-s)^{(\alpha-1)} \leq M, s = 0, 1, 2, \dots, t_1.$$

所以有:

$$|\Phi(t)| = t^{1-\alpha} t^{(\alpha-1)} |b_1| \leq M |b_1|, \quad t \geq t_2. \tag{27}$$

$$\begin{aligned} |\Psi(t, t_1)| &\leq \sum_{s=0}^{t_1-1} t^{1-\alpha} (t-s-1)^{(\alpha-1)} \left| \frac{r(0)}{r(s)} b_0 + \frac{1}{r(s)} \sum_{\xi=0}^{s-1} (-f(z(\xi))q(\xi) + v(\xi)) \right| \leq \\ &M \sum_{s=0}^{t_1-1} t^{1-\alpha} (t-s-1)^{(\alpha-1)} \left| \frac{r(0)}{r(s)} b_0 + \frac{1}{r(s)} \sum_{\xi=0}^{s-1} (-f(z(\xi))q(\xi) + v(\xi)) \right| = \\ &M \sum_{s=0}^{t_1-1} \left(\frac{t}{t-s-1}\right)^{1-\alpha} \left| \frac{r(0)}{r(s)} b_0 + \frac{1}{r(s)} \sum_{\xi=0}^{s-1} (-f(z(\xi))q(\xi) + v(\xi)) \right| \leq \\ &M \sum_{s=0}^{t_1-1} \left(\frac{t_2}{t_2-s-1}\right)^{1-\alpha} \left| \frac{r(0)}{r(s)} b_0 + \frac{1}{r(s)} \sum_{\xi=0}^{s-1} (-f(z(\xi))q(\xi) + v(\xi)) \right| = C(t_1, t_2). \end{aligned} \tag{28}$$

由式(26), 式(27) 和式(28) 得:

$$\begin{aligned} \sum_{s=t_1}^{t-\alpha} t^{1-\alpha} (t-s-1)^{(\alpha-1)} \left[\frac{r(0)}{r(s)} b_0 + \frac{K}{r(s)} \sum_{\xi=0}^{s-1} q(\xi) \sum_{\eta=0}^{\xi-1} (\xi-\eta-1)^{(\alpha-1)} M_2 + \frac{1}{r(s)} \sum_{\xi=0}^{s-1} v(\xi) \right] \geq \\ - (M |b_1| + C(t_1, t_2)), \quad t \geq t_2, \end{aligned}$$

上式两边分别令 $t \rightarrow \infty$ 取极限, 则与式(22) 矛盾。

ii) $\alpha > 1$ 即 $m \geq 2$ 。利用 STIRLING 公式, 得到 $\lim_{t \rightarrow \infty} t^{\alpha-1} t^{(1-\alpha)} = 1$, 即 $\frac{1}{2} < \frac{t^{(1-\alpha)}}{t^{1-\alpha}} < \frac{3}{2}, t \geq t_2$,

运用引理 3 和引理 4, 得到:

$$\begin{aligned} |\Phi(t)| &= \left| t^{1-\alpha} \Gamma(\alpha) \sum_{k=0}^{m-1} \frac{b_{m-k} t^{(\alpha-m+k)}}{\Gamma(\alpha-m+k+1)} \right| \leq t^{1-\alpha} \Gamma(\alpha) \sum_{k=0}^{m-1} \frac{|b_{m-k}| t^{(\alpha-m+k)}}{\Gamma(\alpha-m+k+1)} \leq \\ &\frac{\Gamma(\alpha) |b_m|}{\Gamma(\alpha-m+1)} t^{1-\alpha} t^{(\alpha-m)} + 2\Gamma(\alpha) \sum_{k=1}^{m-1} \frac{|b_{m-k}| t^{(\alpha-m+k)}}{\Gamma(\alpha-m+k+1)} t^{(1-\alpha)} \leq \\ &\frac{M\Gamma(\alpha) |b_m|}{\Gamma(\alpha-m+1)} t_2^{1-m} + 2\Gamma(\alpha) \sum_{k=1}^{m-1} \frac{|b_{m-k}|}{\Gamma(\alpha-m+k+1)} t^{(\alpha-m+k)} [t - (\alpha-m+k)]^{(1-\alpha)} \leq \\ &\frac{M\Gamma(\alpha) |b_m|}{\Gamma(\alpha-m+1)} t_2^{1-m} + 2\Gamma(\alpha) \sum_{k=1}^{m-1} \frac{|b_{m-k}|}{\Gamma(\alpha-m+k+1)} t^{(1-m+k)} \leq \\ &\frac{M\Gamma(\alpha) |b_m|}{\Gamma(\alpha-m+1)} t_2^{1-m} + 2\Gamma(\alpha) \sum_{k=1}^{m-1} \frac{|b_{m-k}|}{\Gamma(\alpha-m+k+1)} t_2^{(1-m+k)} = C_1(t_2). \end{aligned} \tag{29}$$

$$\begin{aligned} |\Psi(t, t_1)| &\leq \sum_{s=0}^{t_1-1} t^{1-\alpha} (t-s-1)^{(\alpha-1)} \left| \frac{r(0)}{r(s)} b_0 + \frac{1}{r(s)} \sum_{\xi=0}^{s-1} (-f(z(\xi))q(\xi) + v(\xi)) \right| \leq \\ &2 \sum_{s=0}^{t_1-1} t^{(1-\alpha)} (t-s-1)^{(\alpha-1)} \left| \frac{r(0)}{r(s)} b_0 + \frac{1}{r(s)} \sum_{\xi=0}^{s-1} (-f(z(\xi))q(\xi) + v(\xi)) \right| \leq \\ &2 \sum_{s=0}^{t_1-1} (t - (\alpha-1))^{(1-\alpha)} t^{(\alpha-1)} \left| \frac{r(0)}{r(s)} b_0 + \frac{1}{r(s)} \sum_{\xi=0}^{s-1} (-f(z(\xi))q(\xi) + v(\xi)) \right| \leq \\ &2 \sum_{s=0}^{t_1-1} \left| \frac{r(0)}{r(s)} b_0 + \frac{1}{r(s)} \sum_{\xi=0}^{s-1} (-f(z(\xi))q(\xi) + v(\xi)) \right| = C_2(t_1). \end{aligned} \tag{30}$$

由式(26), 式(29) 和式(30), 得到:

$$\begin{aligned} \sum_{s=t_1}^{t-\alpha} t^{1-\alpha} (t-s-1)^{(\alpha-1)} \left[\frac{r(0)}{r(s)} b_0 + \frac{K}{r(s)} \sum_{\xi=0}^{s-1} q(\xi) \sum_{\eta=0}^{\xi-1} (\xi-\eta-1)^{(\alpha-1)} M_2 + \frac{1}{r(s)} \sum_{\xi=0}^{s-1} v(\xi) \right] \geq \\ - (C_1(t_2) + C_2(t_1)), \quad t \geq t_2, \end{aligned}$$

上式两边分别令 $t \rightarrow \infty$, 则与式(22) 矛盾。

最后, 假设 $x(t)$ 是方程(2) 的最终负解, 类似可证与式(23) 相矛盾。在此省略, 定理得证。

3 应用

考虑以下 Riemann-Liouville 型分数阶差分方程:

$$\Delta[t^{-\frac{1}{3}}(\Delta^{\frac{1}{5}}x(t))^{\frac{1}{3}}] + \Gamma(\frac{4}{5})t^2\Delta^{-\frac{4}{5}}x(t+1) = 0, \quad t \in N_0, \quad (31)$$

在方程(31)中, $\alpha = \frac{1}{5}$, $r(t) = t^{-\frac{1}{3}}$, $q(t) = t^2$, $\eta = \frac{1}{3}$ 和 $f(z(t)) = z(t+1)$, 得到 $K = 1$ 。

显然: $\sum_{s=0}^{\infty} r^{-\frac{1}{7}}(s) = \sum_{s=0}^{\infty} t = \infty$, 则满足条件(14)。进一步有 $\sum_{s=0}^{\infty} q(s) = \sum_{s=0}^{\infty} s^2 = \infty$ 。

应用定理 1, 得到方程(31)的每个解都是振动的。

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