第 37 卷第 6 期 河北科技大学学报 Vol. 37, No. 6
2016 年 12 月 Journal of Hebei University of Science and Technology Dec. 2016

文章编号:1008-1542(2016)06-0562-13 doi:10.7535/hbkd.2016yx06007

分数阶脉冲微分方程边值问题解的存在性

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摘 要: 为了解决对半无穷区间上具有可数个脉冲点且带有积分边界条件的分数阶脉冲微分方程边值问题,具体研究此类微分方程边值问题解的存在性。通过定义合适的 Banach 空间、范数以及算子,合理运用分数阶微积分的性质,分别应用压缩映像原理和 Krasnoselskii 不动点定理证明了分数阶脉冲微分方程边值问题解的存在性,最后通过实例验证了此类方程边值问题解的存在性。

关键词: 常微分方程解析理论;脉冲;压缩映像原理;Krasnoselskii 不动点定理;边值问题;半无穷区间

中图分类号: O175.8 **MSC(2010)主题分类:** 34B18 **文献标志码:** A

Existence of solutions to boundary value problem of fractional differential equations with impulsive

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Abstract: In order to solve the boundary value problem of fractional impulsive differential equations with countable impulses and integral boundary conditions on the half line, the existence of solutions to the boundary problem is specifically studied. By defining suitable Banach spaces, norms and operators, using the properties of fractional calculus and applying the contraction mapping principle and Krasnoselskii's fixed point theorem, the existence of solutions for the boundary value problem of fractional impulsive differential equations with countable impulses and integral boundary conditions on the half line is proved, and examples are given to illustrate the existence of solutions to this kind of equation boundary value problems.

Keywords: analytic theory of ordinary differential equation; impulse; contraction mapping theorem; Krasnoselskii's fixed point theorem; boundary value problem; the half line

1 问题提出

分数阶微积分是对整数阶微积分理论的拓展,它可以更好地描述某些客观事物或规律,应用广泛,比如在处理光学和热学系统、流变学及材料和力学系统、信号处理和系统辨识、控制等问题的过程中,经常会用到分数阶微积分的理论。所以分数阶微积分理论受到了人们越来越多的关注[1-12]。此外,脉冲微分方程也有广泛的应用,许多学者对脉冲微分方程的理论及其应用[13-24]进行了深入的研究。

文献[2]中 GUO 应用 Banach 空间中的锥拉伸与压缩不动点定理研究了半无穷区间上具有可数个脉冲

收稿日期:2016-03-24;修回日期:2016-09-10;责任编辑:张 军

基金项目:河北省自然科学基金(A2013208108)

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江卫华,李庆敏,周彩莲.分数阶脉冲微分方程边值问题解的存在性[J].河北科技大学学报,2016,37(6):562-574.

JIANG Weihua, LI Qingmin, ZHOU Cailian. Existence of solutions to boundary value problem of fractional differential equations with impulsive[J]. Journal of Hebei University of Science and Technology, 2016, 37(6): 562-574.

点的二阶奇异脉冲微分方程边值问题：

$$\begin{cases} u''(t) = f(t,u(t),u'(t),(Tu)(t),(Su)(t)), \quad t \in R'_{++}, \\ \Delta u|_{t=t_k} = I_k(u'(t_k^-)), \quad k = 1,2,3,\cdots, \\ \Delta u'|_{t=t_k} = \bar{I}_k(u'(t_k^-)), \quad k = 1,2,3,\cdots, \\ u(0) = 0, u'(\infty) = \beta u'(0) \end{cases}$$

解的存在性。

文献[4]中 AHMAD 等根据压缩映像原理和 Krasnoselskii 不动点定理研究了有限区间上具有有限个脉冲点的非线性分数阶脉冲微分方程边值问题：

$$\begin{cases} {}^cD_{0+}^q x(t) = f(t,x(t)), \quad 1 < q < 2, t \in [0,1]\backslash\{t_1,t_2,\cdots,t_p\}, \\ \Delta x(t_k) = I_k(x(t_k^-)), \Delta x'(t_k) = J_k(x(t_k^-)), \quad t_k \in (0,1), k = 1,2,\cdots,p, \\ \alpha x(0) + \beta x'(0) = \int_0^1 q_1(x(s))\mathrm{d}s, \alpha x(1) + \beta x'(1) = \int_0^1 q_2(x(s))\mathrm{d}s \end{cases}$$

解的存在性。

受上述文献的启发，本文将应用压缩映像原理和 Krasnoselskii 不动点定理研究半无穷区间上具有可数个脉冲点的分数阶脉冲微分方程边值问题：

$$\begin{cases} D_{0+}^\alpha u(t) = f(t,u(t),D_{0+}^{\alpha-1}u(t)), \\ \Delta D_{0+}^{\alpha-1}u|_{t=t_k} = I_k(u(t_k^-)), k = 1,2,\cdots, \\ u(0) = 0, D_{0+}^{\alpha-1}u(\infty) = \int_0^\infty h(t)D_{0+}^{\alpha-1}u(t)\mathrm{d}t \end{cases} \tag{1}$$

解的存在性，其中 $D_{0+}^\alpha u(t)$ 是 $u(t)$ 的 α 阶 Riemann-Liuville 导数，$1<\alpha<2$。令 $J=[0,+\infty)$，$J_1=[0,t_1]$，$J_k=(t_{k-1},t_k]$，$k=2,3,\cdots,0<t_1<t_2<\cdots<t_k<\cdots$。$f\in C[J\times\mathbf{R}^2,\mathbf{R}]$，$I_k\in C[\mathbf{R},\mathbf{R}]$。$\Delta D_{0+}^{\alpha-1}u|_{t=t_k} = D_{0+}^{\alpha-1}u(t_k^+) - D_{0+}^{\alpha-1}u(t_k^-)$，其中 $D_{0+}^{\alpha-1}u(t_k^+) = \lim\limits_{h\to 0^+} D_{0+}^{\alpha-1}u(t_k+h)$，$D_{0+}^{\alpha-1}u(t_k^-) = \lim\limits_{h\to 0^-} D_{0+}^{\alpha-1}u(t_k+h)$。$h(t)\in L^1[0,+\infty)$，$1<\int_0^\infty h(t)\mathrm{d}t<2$，$h(t)\geqslant 0$。

2　预备知识

定义 1　$u: J\to\mathbf{R}$ 是连续函数，u 的 α 阶 Riemann-Liuville 积分的定义式为

$$I_{0+}^\alpha u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\mathrm{d}s, \quad \alpha > 0。$$

定义 2　$u: J\to\mathbf{R}$ 是连续函数，u 的 α 阶 Riemann-Liuville 导数的定义式为

$$D_{0+}^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_0^t (t-s)^{n-\alpha-1}u(s)\mathrm{d}s, n = [\alpha]+1, \alpha > 0。$$

引理 1　$D_{0+}^\alpha u(t) = 0$ 当且仅当 $u(t) = c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n}, n = [\alpha]+1, c_i \in \mathbf{R}, i = 1,2,\cdots,n$。

引理 2　$I_{0+}^\alpha D_{0+}^\alpha u(t) = u(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n}, n = [\alpha]+1, c_i \in \mathbf{R}, i = 1,2,\cdots,n$。

引理 3　若 $\alpha > 0, \lambda > -1$，则 $D_{0+}^\alpha t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}t^{\lambda-\alpha}$。

定理 1　（压缩映像原理）

设 X 是完备的度量空间，T 是 X 上的压缩映像，那么 T 有且仅有 1 个不动点。

定理 2　（Krasnoselskii 不动点定理）

设 M 是 Banach 空间 X 中的一个非空凸闭子集。假设 A,B 是 2 个算子，满足：

a）对任意的 $x,y\in M$，有 $Ax+By\in M$；

b）A 是全连续映射；

c）B 是一个压缩映射，

则至少存在一个 $z\in M$，使得 $z=Az+Bz$。

定义空间 $PC[J,\mathbf{R}]=\{u: u\in C[J,\mathbf{R}], D_{0+}^{\alpha-1}u\in C(J_k), D_{0+}^{\alpha-1}u(t_k^+), D_{0+}^{\alpha-1}u(t_k^-)\text{存在}, k=1,2,\cdots\}$。$PC^1[J,\mathbf{R}]=\left\{u\in PC[J,\mathbf{R}]: \sup\limits_{t\in J}\frac{|u(t)|}{1+t^{\alpha-1}}<\infty, \sup\limits_{t\in J}|D_{0+}^{\alpha-1}u(t)|<\infty\right\}$，$u$ 的范数定义为 $\|u\|=\max\{\|u\|_S, \|D_{0+}^{\alpha-1}u\|_B\}$，其中 $\|u\|_S=\sup\limits_{t\in J}\frac{|u(t)|}{1+t^{\alpha-1}}$，$\|D_{0+}^{\alpha-1}u\|_B=\sup\limits_{t\in J}|D_{0+}^{\alpha-1}u(t)|$。显然，$PC^1[J,\mathbf{R}]$是 Banach 空间。

引理 4 对于给定的函数 $y\in C(J_k)$，$k=1,2,\cdots$，$u(t)$是分数阶脉冲微分方程边值问题：

$$\begin{cases} D_{0+}^{\alpha}u(t)=y(t), \\ \Delta D_{0+}^{\alpha-1}u\mid_{t=t_k}=I_k(u(t_k^-)),k=1,2,\cdots, \\ u(0)=0,D_{0+}^{\alpha-1}u(\infty)=\int_0^{\infty}h(t)D_{0+}^{\alpha-1}u(t)\mathrm{d}t \end{cases} \tag{2}$$

的解当且仅当 $u(t)$满足

$$u(t)=\frac{1}{\Gamma(\alpha)}\Big[\int_0^t(t-s)^{\alpha-1}y(s)\mathrm{d}s+\sum_{0<t_k<t}I_k(u(t_k^-))(t-t_k)^{\alpha-1}+ \frac{t^{\alpha-1}}{\int_0^{\infty}h(t)\mathrm{d}t-1}\Big(\int_0^{\infty}y(s)(1-\int_s^{\infty}h(t)\mathrm{d}t)\mathrm{d}s+\sum_{k=1}^{\infty}I_k(u(t_k^-))(1-\int_{t_k}^{\infty}h(t)\mathrm{d}t)\Big)\Big]。\tag{3}$$

证明 设 $u(t)$ 是分数阶脉冲微分方程边值问题(2) 的解，由引理 2 可得当 $t\in[0,t_1]$ 时，

$$u(t)=I_{0+}^{\alpha}y(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2},$$

因为边值条件 $u(0)=0$，所以 $c_2=0$，即 $u(t)=I_{0+}^{\alpha}y(t)+c_1t^{\alpha-1}=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\mathrm{d}s+c_1t^{\alpha-1}$。

所以 $u(t_1^-)=\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}y(s)\mathrm{d}s+c_1t_1^{\alpha-1}$。由引理 3 可得

$$D_{0+}^{\alpha-1}u(t)=D_{0+}^{\alpha-1}(I_{0+}^{\alpha}y(t)+c_1t^{\alpha-1})=\int_0^t y(s)\mathrm{d}s+c_1\Gamma(\alpha)。$$

当 $t\in(t_1,t_2]$ 时，因为 $\Delta D_{0+}^{\alpha-1}u\mid_{t=t_1}=I_1(u(t_1^-))$，所以 $D_{0+}^{\alpha-1}u(t)=\int_0^t y(s)\mathrm{d}s+c_1\Gamma(\alpha)+I_1(u(t_1^-))$，

$$\begin{aligned} u(t)&=\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}D_{0+}^{\alpha-1}u(s)\mathrm{d}s+b_1t^{\alpha-2}= \\ &\frac{1}{\Gamma(\alpha-1)}\Big\{\int_0^{t_1}(t-s)^{\alpha-2}\Big(\int_0^s y(r)\mathrm{d}r+c_1\Gamma(\alpha)\Big)\mathrm{d}s+ \\ &\int_{t_1}^t(t-s)^{\alpha-2}\Big(\int_0^s y(r)\mathrm{d}r+c_1\Gamma(\alpha)+I_1(u(t_1^-))\Big)\mathrm{d}s\Big\}+b_1t^{\alpha-2}= \\ &\frac{1}{\Gamma(\alpha)}\Big[\int_0^t(t-s)^{\alpha-1}y(s)\mathrm{d}s+c_1\Gamma(\alpha)t^{\alpha-1}+I_1(u(t_1^-))(t-t_1)^{\alpha-1}\Big]+b_1t^{\alpha-2},\quad b_1\in\mathbf{R}, \end{aligned}$$

所以 $u(t_1^+)=\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}y(s)\mathrm{d}s+c_1t_1^{\alpha-1}+b_1t_1^{\alpha-2}$，又因为 $u(t_1^+)=u(t_1^-)$，所以 $b_1=0$。因此 $t\in(t_1,t_2]$ 时，$u(t)=\frac{1}{\Gamma(\alpha)}\Big[\int_0^t(t-s)^{\alpha-1}y(s)\mathrm{d}s+c_1\Gamma(\alpha)t^{\alpha-1}+I_1(u(t_1^-))(t-t_1)^{\alpha-1}\Big]$。

依此类推可得，当 $t\in J_k=(t_{k-1},t_k]$ 时，$D_{0+}^{\alpha-1}u(t)=\int_0^t y(s)\mathrm{d}s+c_1\Gamma(\alpha)+\sum_{i=1}^{k-1}I_i(u(t_i^-))$，

$$\begin{aligned} u(t)&=\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}D_{0+}^{\alpha-1}u(s)\mathrm{d}s+bt^{\alpha-2}= \\ &\frac{1}{\Gamma(\alpha-1)}\Big[\int_0^{t_1}(t-s)^{\alpha-2}\Big(\int_0^s y(r)\mathrm{d}r+c_1\Gamma(\alpha)\Big)\mathrm{d}s+ \\ &\int_{t_1}^{t_2}(t-s)^{\alpha-2}\Big(\int_0^s y(r)\mathrm{d}r+c_1\Gamma(\alpha)+I_1(u(t_1^-))\Big)\mathrm{d}s+\cdots+ \\ &\int_{t_{k-1}}^t(t-s)^{\alpha-2}\Big(\int_0^s y(r)\mathrm{d}r+c_1\Gamma(\alpha)+\sum_{i=1}^{k-1}I_i(u(t_i^-))\Big)\mathrm{d}s\Big]+bt^{\alpha-2}= \\ &\frac{1}{\Gamma(\alpha-1)}\Big[\int_0^t(t-s)^{\alpha-2}\Big(\int_0^s y(r)\mathrm{d}r+c_1\Gamma(\alpha)\Big)\mathrm{d}s+ \\ &\int_{t_1}^t(t-s)^{\alpha-2}I_1(u(t_1^-))\mathrm{d}s+\int_{t_2}^t(t-s)^{\alpha-2}I_2(u(t_2^-))\mathrm{d}s+\cdots+ \\ &\int_{t_{k-1}}^t(t-s)^{\alpha-2}I_{k-1}(u(t_{k-1}^-))\mathrm{d}s\Big]+bt^{\alpha-2}= \\ &\frac{1}{\Gamma(\alpha)}\Big[\int_0^t(t-s)^{\alpha-1}y(s)\mathrm{d}s+c_1\Gamma(\alpha)t^{\alpha-1}+\sum_{i=1}^{k-1}I_i(u(t_i^-))(t-t_i)^{\alpha-1}\Big]+bt^{\alpha-2},\quad b\in\mathbf{R}。 \end{aligned}$$

同理由 $u(t)$ 的连续性可知 $b=0$，所以 $t\in J_k=(t_{k-1},t_k]$ 时，

$$u(t)=\frac{1}{\Gamma(\alpha)}\Big[\int_0^t(t-s)^{\alpha-1}y(s)\mathrm{d}s+c_1\Gamma(\alpha)t^{\alpha-1}+\sum_{i=1}^{k-1}I_i(u(t_i^-))(t-t_i)^{\alpha-1}\Big]。$$

因此，对 $\forall t \in J$ 有

$$u(t)=\frac{1}{\Gamma(\alpha)}\Big[\int_0^t (t-s)^{\alpha-1}y(s)\mathrm{d}s+c_1\Gamma(\alpha)t^{\alpha-1}+\sum_{0<t_k<t}I_k(u(t_k^-))(t-t_k)^{\alpha-1}\Big], \tag{4}$$

$$D_{0+}^{\alpha-1}u(t)=\int_0^t y(s)\mathrm{d}s+c_1\Gamma(\alpha)+\sum_{0<t_k<t}I_k(u(t_k^-))。$$

因为 $D_{0+}^{\alpha-1}u(\infty)=\int_0^\infty h(t)D_{0+}^{\alpha-1}u(t)\mathrm{d}t$，所以

$$\int_0^\infty y(s)\mathrm{d}s+c_1\Gamma(\alpha)+\sum_{k=1}^\infty I_k(u(t_k^-))=$$
$$\int_0^\infty h(t)\Big(\int_0^t y(s)\mathrm{d}s+c_1\Gamma(\alpha)+\sum_{0<t_k<t}I_k(u(t_k^-))\Big)\mathrm{d}t=$$
$$\int_0^\infty y(s)\int_s^\infty h(t)\mathrm{d}t\mathrm{d}s+c_1\Gamma(\alpha)\int_0^\infty h(t)\mathrm{d}t+\int_{t_1}^{t_2}h(t)I_1(u(t_1^-))\mathrm{d}t+$$
$$\int_{t_2}^{t_3}h(t)(I_1(u(t_1^-))+I_2(u(t_2^-)))\mathrm{d}t+\cdots+\int_{t_{k-1}}^{t_k}h(t)\sum_{i=1}^{k-1}I_i(u(t_i^-))\mathrm{d}t+\cdots=$$
$$\int_0^\infty y(s)\int_s^\infty h(t)\mathrm{d}t\mathrm{d}s+c_1\Gamma(\alpha)\int_0^\infty h(t)\mathrm{d}t+\sum_{k=1}^\infty I_k(u(t_k^-))\int_{t_k}^\infty h(t)\mathrm{d}t,$$

解得 $c_1=\dfrac{1}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}\Big[\int_0^\infty y(s)(1-\int_s^\infty h(t)\mathrm{d}t)\mathrm{d}s+\sum_{k=1}^\infty I_k(u(t_k^-))(1-\int_{t_k}^\infty h(t)\mathrm{d}t)\Big]$。把 c_1 代入式(4)中便可得式(3)。反之，式(3) 经过求导计算可得到式(2)。

3　主要结果

设 $f:J\times\mathbf{R}^2\to\mathbf{R}$ 是连续函数，$I_k\in C[\mathbf{R},\mathbf{R}]$，$k=1,2,\cdots$，$f(t,u,\tilde{u})$ 和 $I_k(u)$ 满足以下条件：

H_1）存在函数 $a,b\in C[J,J]$，使得对 $\forall t\in J$，$u,\tilde{u},v,\tilde{v}\in\mathbf{R}$，有

$$|f(t,u,\tilde{u})-f(t,v,\tilde{v})|\leqslant a(t)\frac{|u-v|}{1+t^{\alpha-1}}+b(t)|\tilde{u}-\tilde{v}|,$$

且 $a^*=\int_0^\infty a(t)\mathrm{d}t<1$，$b^*=\int_0^\infty b(t)\mathrm{d}t<1$。

H_2）存在常数 $\gamma_k\in J$，使得对 $\forall t\in J$，$u,v\in\mathbf{R}$，有

$$|I_k(u)-I_k(v)|\leqslant\gamma_k\|u-v\|,$$

且 $\gamma^*=\sum_{k=1}^\infty\gamma_k<1$。

H_3）存在函数 $c,d\in C[J,J]$，$g,h\in C[\mathbf{R},J]$，使得对 $\forall t\in J$，$u,\tilde{u}\in\mathbf{R}$，有

$$|f(t,u,\tilde{u})|\leqslant c(t)g\Big(\frac{u}{1+t^{\alpha-1}}\Big)+d(t)h(\tilde{u}),$$

且 $c^*=\int_0^\infty c(t)\mathrm{d}t<\infty$，$d^*=\int_0^\infty d(t)\mathrm{d}t<\infty$。

H_4）存在函数 $F\in C[\mathbf{R},J]$，常数 $\eta_k\in J$，使得对 $\forall t\in J$，$u\in\mathbf{R}$，有

$$|I_k(u)|\leqslant\eta_kF(u),$$

且 $\eta^*=\sum_{k=1}^\infty\eta_k<\infty$。

定理 3　假设条件 H_1）—条件 H_2）成立，且满足条件 $\rho=\dfrac{(a^*+b^*+\gamma^*)\int_0^\infty h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}<1$，则边值问题(1)存在唯一解。

证明　定义算子 $T:PC^1[J,\mathbf{R}]\to PC^1[J,\mathbf{R}]$ 如下：

$$(Tu)(t)=\frac{1}{\Gamma(\alpha)}\Big[\int_0^t(t-s)^{\alpha-1}f(s,u(s),D_{0+}^{\alpha-1}u(s))\mathrm{d}s+\sum_{0<t_k<t}I_k(u(t_k^-))(t-t_k)^{\alpha-1}+$$
$$\frac{t^{\alpha-1}}{\int_0^\infty h(t)\mathrm{d}t-1}\Big(\int_0^\infty f(s,u(s),D_{0+}^{\alpha-1}u(s))(1-\int_s^\infty h(t)\mathrm{d}t)\mathrm{d}s+\sum_{k=1}^\infty I_k(u(t_k^-))(1-\int_{t_k}^\infty h(t)\mathrm{d}t)\Big)\Big],$$

所以

$$D_{0+}^{\alpha-1}(Tu)(t)=\int_0^t f(s,u(s),D_{0+}^{\alpha-1}u(s))\mathrm{d}s+\sum_{0<t_k<t}I_k(u(t_k^-))+$$
$$\frac{1}{\int_0^\infty h(t)\mathrm{d}t-1}\left(\int_0^\infty f(s,u(s),D_{0+}^{\alpha-1}u(s))(1-\int_s^\infty h(t)\mathrm{d}t)\mathrm{d}s+\sum_{k=1}^\infty I_k(u(t_k^-))(1-\int_{t_k}^\infty h(t)\mathrm{d}t)\right)。$$

对 $\forall u,v\in PC^1[J,\mathbf{R}],\forall t\in J$ 有

$$\frac{|(Tu)(t)-(Tv)(t)|}{1+t^{\alpha-1}}\leqslant$$
$$\frac{1}{\Gamma(\alpha)}\Big[\int_0^t\frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}\,|f(s,u(s),D_{0+}^{\alpha-1}u(s))-f(s,v(s),D_{0+}^{\alpha-1}v(s))|\,\mathrm{d}s+$$
$$\frac{t^{\alpha-1}}{(1+t^{\alpha-1})(\int_0^\infty h(t)\mathrm{d}t-1)}\Big(\int_0^\infty|f(s,u(s),D_{0+}^{\alpha-1}u(s))-f(s,v(s),D_{0+}^{\alpha-1}v(s))|\cdot$$
$$|1-\int_s^\infty h(t)\mathrm{d}t|\,\mathrm{d}s+\sum_{k=1}^\infty|I_k(u(t_k^-))-I_k(v(t_k^-))|\cdot$$
$$|1-\int_{t_k}^\infty h(t)\mathrm{d}t|\Big)+\sum_{0<t_k<t}|I_k(u(t_k^-))-I_k(v(t_k^-))|\,\frac{(t-t_k)^{\alpha-1}}{1+t^{\alpha-1}}\Big]\leqslant$$
$$\frac{\int_0^\infty h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}\Big[\int_0^\infty|f(s,u(s),D_{0+}^{\alpha-1}u(s))-f(s,v(s),D_{0+}^{\alpha-1}v(s))|\,\mathrm{d}s+$$
$$\sum_{k=1}^\infty|I_k(u(t_k^-))-I_k(v(t_k^-))|\Big]\leqslant$$
$$\frac{\int_0^\infty h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}\Big[\int_0^\infty\Big(a(s)\frac{|u(s)-v(s)|}{1+s^{\alpha-1}}+b(s)\,|D_{0+}^{\alpha-1}u(s)-D_{0+}^{\alpha-1}v(s)|\Big)\mathrm{d}s+$$
$$\sum_{k=1}^\infty\gamma_k\|u-v\|\Big]\leqslant$$
$$\frac{\int_0^\infty h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}\Big[\int_0^\infty(a(s)\|u-v\|_S+b(s)\|D_{0+}^{\alpha-1}u-D_{0+}^{\alpha-1}v\|_B)\mathrm{d}s+$$
$$\sum_{k=1}^\infty\gamma_k\|u-v\|\Big]\leqslant$$
$$\frac{\int_0^\infty h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}\Big[\int_0^\infty a(s)\mathrm{d}s+\int_0^\infty b(s)\mathrm{d}s+\sum_{k=1}^\infty\gamma_k\Big]\|u-v\|=$$
$$\frac{(a^*+b^*+\gamma^*)\int_0^\infty h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}\|u-v\|,$$

所以 $\|Tu-Tv\|_S\leqslant\rho\|u-v\|$。

$$|D_{0+}^{\alpha-1}(Tu)(t)-D_{0+}^{\alpha-1}(Tv)(t)|\leqslant$$
$$\int_0^t|f(s,u(s),D_{0+}^{\alpha-1}u(s))-f(s,v(s),D_{0+}^{\alpha-1}v(s))|\,\mathrm{d}s+$$
$$\frac{1}{\int_0^\infty h(t)\mathrm{d}t-1}\Big[\int_0^\infty|f(s,u(s),D_{0+}^{\alpha-1}u(s))-f(s,v(s),D_{0+}^{\alpha-1}v(s))|\cdot$$
$$|1-\int_s^\infty h(t)\mathrm{d}t|\,\mathrm{d}s+\sum_{k=1}^\infty|I_k(u(t_k^-))-I_k(v(t_k^-))|\cdot$$
$$|1-\int_{t_k}^\infty h(t)\mathrm{d}t|\Big]+\sum_{0<t_k<t}|I_k(u(t_k^-))-I_k(v(t_k^-))|\leqslant$$

$$\frac{\int_0^\infty h(t)\mathrm{d}t}{\int_0^\infty h(t)\mathrm{d}t-1}\Big[\int_0^\infty |f(s,u(s),D_{0+}^{\alpha-1}u(s))-f(s,v(s),D_{0+}^{\alpha-1}v(s))|\mathrm{d}s+$$

$$\sum_{k=1}^\infty |I_k(u(t_k^-))-I_k(v(t_k^-))|\Big]\leqslant$$

$$\frac{\int_0^\infty h(t)\mathrm{d}t}{\int_0^\infty h(t)\mathrm{d}t-1}\Big[\int_0^\infty \Big(a(s)\frac{|u(s)-v(s)|}{1+s^{\alpha-1}}+b(s)|D_{0+}^{\alpha-1}u(s)-D_{0+}^{\alpha-1}v(s)|\Big)\mathrm{d}s+$$

$$\sum_{k=1}^\infty \gamma_k\|u-v\|\Big]\leqslant\frac{\int_0^\infty h(t)\mathrm{d}t}{\int_0^\infty h(t)\mathrm{d}t-1}\Big[\int_0^\infty a(s)\mathrm{d}s+\int_0^\infty b(s)\mathrm{d}s+\sum_{k=1}^\infty\gamma_k\Big]\|u-v\|=$$

$$\frac{(a^*+b^*+\gamma^*)\int_0^\infty h(t)\mathrm{d}t}{\int_0^\infty h(t)\mathrm{d}t-1}\|u-v\|,$$

因为 $1<\alpha<2$ 时,$0<\Gamma(\alpha)<1$,所以 $\|D_{0+}^{\alpha-1}(Tu)-D_{0+}^{\alpha-1}(Tv)\|_B<\rho\|u-v\|$。所以 $\|Tu-Tv\|\leqslant\rho\|u-v\|$。又因为 $\rho<1$,所以 T 是一个压缩算子。因此,根据压缩映像原理可知,边值问题(1)有唯一解。

定理 4　假设条件 H_2)— 条件 H_4)成立,且满足 $\dfrac{\gamma^*\int_0^\infty h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}<1$,则边值问题(1)至少有 1 个解。

证明　定义算子如下:

$$(Au)(t)=\frac{1}{\Gamma(\alpha)}\Big[\int_0^t(t-s)^{\alpha-1}f(s,u(s),D_{0+}^{\alpha-1}u(s))\mathrm{d}s+$$

$$\frac{t^{\alpha-1}}{\int_0^\infty h(t)\mathrm{d}t-1}\int_0^\infty f(s,u(s),D_{0+}^{\alpha-1}u(s))(1-\int_s^\infty h(t)\mathrm{d}t)\mathrm{d}s\Big],$$

$$(Bu)(t)=\frac{1}{\Gamma(\alpha)}\left[\frac{t^{\alpha-1}}{\int_0^\infty h(t)\mathrm{d}t-1}\sum_{k=1}^\infty I_k(u(t_k^-))(1-\int_{t_k}^\infty h(t)\mathrm{d}t)+\sum_{0<t_k<t}I_k(u(t_k^-))(t-t_k)^{\alpha-1}\right]。$$

所以

$$D_{0+}^{\alpha-1}(Au)(t)=\int_0^t f(s,u(s),D_{0+}^{\alpha-1}u(s))\mathrm{d}s+\frac{1}{\int_0^\infty h(t)\mathrm{d}t-1}\int_0^\infty f(s,u(s),D_{0+}^{\alpha-1}u(s))(1-\int_s^\infty h(t)\mathrm{d}t)\mathrm{d}s,$$

$$D_{0+}^{\alpha-1}(Bu)(t)=\frac{1}{\int_0^\infty h(t)\mathrm{d}t-1}\sum_{k=1}^\infty I_k(u(t_k^-))(1-\int_{t_k}^\infty h(t)\mathrm{d}t)+\sum_{0<t_k<t}I_k(u(t_k^-))。$$

由条件 H_3)— 条件 H_4)可知:对 $\forall r>0$,

$$|f(t,u,\tilde{u})|\leqslant c(t)g\Big(\frac{u}{1+t^{\alpha-1}}\Big)+d(t)h(\tilde{u})\leqslant c(t)g_r+d(t)h_r,$$

$$|I_k(u)|\leqslant\eta_kF(u)\leqslant N\eta_k,\quad k=1,2,\cdots,$$

其中 $g_r=\max\Big\{g\Big(\frac{u}{1+t^{\alpha-1}}\Big):\|u\|_S\leqslant r\Big\}$,$h_r=\max\{h(\tilde{u}):\|\tilde{u}\|_B\leqslant r\}$,$N=\max\{F(u):\|u\|\leqslant r\}$。

取 $B_r=\{u\in PC^1[J,\mathbf{R}]:\|u\|\leqslant r\}$,其中 $r\geqslant\dfrac{(c^*g_r+d^*h_r+N\eta^*)\int_0^\infty h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}$。

对 $\forall u,v\in B_r$,$\forall t\in J$,

$$\frac{|(Au)(t)+(Bv)(t)|}{1+t^{\alpha-1}}\leqslant$$

$$\frac{1}{\Gamma(\alpha)}\Big[\int_0^t\frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}|f(s,u(s),D_{0+}^{\alpha-1}u(s))\mathrm{d}s+$$

$$\frac{t^{\alpha-1}}{(1+t^{\alpha-1})(\int_0^\infty h(t)\mathrm{d}t-1)}\int_0^\infty |f(s,u(s),D_{0+}^{\alpha-1}u(s))|\cdot$$

$$|1-\int_s^\infty h(t)\mathrm{d}t|\mathrm{d}s+\frac{t^{\alpha-1}}{(1+t^{\alpha-1})(\int_0^\infty h(t)\mathrm{d}t-1)}\sum_{k=1}^\infty |I_k(v(t_k^-))|\cdot$$

$$|1-\int_{t_k}^\infty h(t)\mathrm{d}t|+\sum_{0<t_k<t}|I_k(v(t_k^-))|\cdot\frac{|t-t_k|^{\alpha-1}}{1+t^{\alpha-1}}\Big]\leqslant$$

$$\frac{\int_0^\infty h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}\Big[\int_0^\infty |f(s,u(s),D_{0+}^{\alpha-1}u(s))|\mathrm{d}s+\sum_{k=1}^\infty |I_k(v(t_k^-))|\Big]\leqslant$$

$$\frac{\int_0^\infty h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}\Big[\int_0^\infty (c(s)g_r+d(s)h_r)\mathrm{d}s+\sum_{k=1}^\infty N\eta_k\Big]=$$

$$\frac{(c^*g_r+d^*h_r+N\eta^*)\int_0^\infty h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}\leqslant r,$$

所以 $\|Au+Bv\|_S\leqslant r$。

$$|D_{0+}^{\alpha-1}(Au)(t)+D_{0+}^{\alpha-1}(Bv)(t)|\leqslant$$

$$\int_0^t |f(s,u(s),D_{0+}^{\alpha-1}u(s))|\mathrm{d}s+\frac{1}{\int_0^\infty h(t)\mathrm{d}t-1}\int_0^\infty |f(s,u(s),D_{0+}^{\alpha-1}u(s))|\cdot$$

$$|1-\int_s^\infty h(t)\mathrm{d}t|\mathrm{d}s+\frac{1}{\int_0^\infty h(t)\mathrm{d}t-1}\sum_{k=1}^\infty |I_k(v(t_k^-))|\cdot$$

$$|1-\int_{t_k}^\infty h(t)\mathrm{d}t|+\sum_{0<t_k<t}|I_k(v(t_k^-))|\leqslant$$

$$\frac{\int_0^\infty h(t)\mathrm{d}t}{\int_0^\infty h(t)\mathrm{d}t-1}\Big[\int_0^\infty |f(s,u(s),D_{0+}^{\alpha-1}u(s))|\mathrm{d}s+\sum_{k=1}^\infty |I_k(v(t_k^-))|\Big]\leqslant$$

$$\frac{\int_0^\infty h(t)\mathrm{d}t}{\int_0^\infty h(t)\mathrm{d}t-1}\Big[\int_0^\infty (c(s)g_r+d(s)h_r)\mathrm{d}s+\sum_{k=1}^\infty N\eta_k\Big]=$$

$$\frac{(c^*g_r+d^*h_r+N\eta^*)\int_0^\infty h(t)\mathrm{d}t}{\int_0^\infty h(t)\mathrm{d}t-1}<r,$$

所以 $\|D_{0+}^{\alpha-1}(Au)+D_{0+}^{\alpha-1}(Bv)\|_B\leqslant r$。所以 $\|Au+Bv\|\leqslant r$。因此,$Au+Bv\in B_r$。

下证 Bu 为压缩算子。对 $\forall u,v\in B_r$,$\forall t\in J$ 有

$$\frac{|(Bu)(t)-(Bv)(t)|}{1+t^{\alpha-1}}\leqslant$$

$$\frac{t^{\alpha-1}}{\Gamma(\alpha)(1+t^{\alpha-1})(\int_0^\infty h(t)\mathrm{d}t-1)}\sum_{k=1}^\infty |I_k(u(t_k^-))-I_k(v(t_k^-))|\cdot$$

$$|1-\int_{t_k}^\infty h(t)\mathrm{d}t|+\frac{1}{\Gamma(\alpha)(1+t^{\alpha-1})}\sum_{0<t_k<t}|I_k(u(t_k^-))-I_k(v(t_k^-))|(t-t_k)^{\alpha-1}\leqslant$$

$$\frac{\int_0^\infty h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}\sum_{k=1}^\infty |I_k(u(t_k^-))-I_k(v(t_k^-))|\leqslant$$

$$\frac{\int_0^\infty h(t)\mathrm{d}t\sum_{k=1}^{\infty}\gamma_k}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}\|u-v\| =$$

$$\frac{\gamma^*\int_0^\infty h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}\|u-v\|,$$

所以 $\|Bu-Bv\|_S \leqslant \frac{\gamma^*\int_0^\infty h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}\|u-v\|$。

$$|D_{0+}^{\alpha-1}(Bu)(t)-D_{0+}^{\alpha-1}(Bv)(t)| \leqslant$$

$$\frac{1}{\int_0^\infty h(t)\mathrm{d}t-1}\sum_{k=1}^{\infty}|I_k(u(t_k^-))-I_k(v(t_k^-))|\cdot$$

$$|1-\int_{t_k}^\infty h(t)\mathrm{d}t|+\sum_{0<t_k<t}|I_k(u(t_k^-))-I_k(v(t_k^-))|\leqslant$$

$$\frac{\int_0^\infty h(t)\mathrm{d}t}{\int_0^\infty h(t)\mathrm{d}t-1}\sum_{k=1}^{\infty}|I_k(u(t_k^-))-I_k(v(t_k^-))|\leqslant$$

$$\frac{\int_0^\infty h(t)\mathrm{d}t\sum_{k=1}^{\infty}\gamma_k}{\int_0^\infty h(t)\mathrm{d}t-1}\|u-v\| <$$

$$\frac{\gamma^*\int_0^\infty h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}\|u-v\|,$$

所以 $\|D_{0+}^{\alpha-1}(Bu)-D_{0+}^{\alpha-1}(Bv)\|_B < \frac{\gamma^*\int_0^\infty h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}\|u-v\|$。

因此，$\|Bu-Bv\| \leqslant \frac{\gamma^*\int_0^\infty h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}\|u-v\|$。又因为 $\frac{\gamma^*\int_0^\infty h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}<1$，所以 Bu 是压缩算子。

下证算子 A 的全连续性。取 $u_n,u\in PC^1[J,\mathbf{R}]$，$\lim_{n\to\infty}\|u_n-u\|=0$，则存在 $r>0$，使得 $\|u_n\|\leqslant r$，$\|u\|\leqslant r$。由 $f(s,u(s),D_{0+}^{\alpha-1}u(s))$ 的连续性可知，对 $\forall t\in J$，当 $n\to\infty$ 时，

$$f(t,u_n(t),D_{0+}^{\alpha-1}u_n(t))\to f(t,u(t),D_{0+}^{\alpha-1}u(t))。$$

由条件 H_3) 可知 $|f(t,u_n(t),D_{0+}^{\alpha-1}u_n(t))-f(t,u(t),D_{0+}^{\alpha-1}u(t))|\leqslant 2(c(t)g_r+d(t)h_r)$。

而 $c^*=\int_0^\infty c(t)\mathrm{d}t<\infty$，$d^*=\int_0^\infty d(t)\mathrm{d}t<\infty$。所以由控制收敛定理可得

$$\lim_{n\to\infty}\int_0^\infty |f(t,u_n(t),D_{0+}^{\alpha-1}u_n(t))-f(t,u(t),D_{0+}^{\alpha-1}u(t))|\mathrm{d}t=0。$$

对 $\forall t\in J$，

$$\frac{|(Au_n)(t)-(Au)(t)|}{1+t^{\alpha-1}}\leqslant$$

$$\frac{1}{\Gamma(\alpha)(1+t^{\alpha-1})}\Big[\int_0^t|t-s|^{\alpha-1}\cdot|f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))-f(s,u(s),D_{0+}^{\alpha-1}u(s))|\mathrm{d}s+$$

$$\frac{t^{\alpha-1}}{\int_0^\infty h(t)\mathrm{d}t-1}\int_0^\infty|f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))-f(s,u(s),D_{0+}^{\alpha-1}u(s))|\cdot|1-\int_s^\infty h(t)\mathrm{d}t|\mathrm{d}s\Big]\leqslant$$

$$\frac{\int_0^{\infty}h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^{\infty}h(t)\mathrm{d}t-1)}\int_0^{\infty}|f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))-f(s,u(s),D_{0+}^{\alpha-1}u(s))|\mathrm{d}s,$$

所以$\lim\limits_{n\to\infty}\|Au_n-Au\|_S=0$。

$$\begin{aligned}&|D_{0+}^{\alpha-1}(Au_n)(t)-D_{0+}^{\alpha-1}(Au)(t)|\leqslant\\&\quad\int_0^t|f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))-f(s,u(s),D_{0+}^{\alpha-1}u(s))|\mathrm{d}s+\\&\quad\frac{1}{\int_0^{\infty}h(t)\mathrm{d}t-1}\int_0^{\infty}|f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))-f(s,u(s),D_{0+}^{\alpha-1}u(s))|\cdot|1-\int_s^{\infty}h(t)\mathrm{d}t|\mathrm{d}s\leqslant\\&\quad\frac{\int_0^{\infty}h(t)\mathrm{d}t}{\int_0^{\infty}h(t)\mathrm{d}t-1}\int_0^{\infty}|f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))-f(s,u(s),D_{0+}^{\alpha-1}u(s))|\mathrm{d}s,\end{aligned}$$

所以$\lim\limits_{n\to\infty}\|D_{0+}^{\alpha-1}Au_n-D_{0+}^{\alpha-1}Au\|_B=0$。故$\lim\limits_{n\to\infty}\|Au_n-Au\|=0$。$Au$ 的连续性得证。

下证算子 A 的紧性。取 $u_n\in B_r=\{u\in PC^1[J,\mathbf{R}]:\|u\|\leqslant r\}$。对 $\forall t\in J$，

$$\begin{aligned}\frac{|(Au_n)(t)|}{1+t^{\alpha-1}}&\leqslant\frac{1}{\Gamma(\alpha)(1+t^{\alpha-1})}\int_0^t(t-s)^{\alpha-1}|f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))|\mathrm{d}s+\\&\quad\frac{t^{\alpha-1}}{\Gamma(\alpha)(1+t^{\alpha-1})(\int_0^{\infty}h(t)\mathrm{d}t-1)}\int_0^{\infty}|f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))|\cdot|1-\int_s^{\infty}h(t)\mathrm{d}t|\mathrm{d}s\leqslant\\&\quad\frac{\int_0^{\infty}h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^{\infty}h(t)\mathrm{d}t-1)}\int_0^{\infty}|f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))|\mathrm{d}s\leqslant\\&\quad\frac{\int_0^{\infty}h(t)\mathrm{d}t\int_0^{\infty}(c(s)g_r+d(s)h_r)\mathrm{d}s}{\Gamma(\alpha)(\int_0^{\infty}h(t)\mathrm{d}t-1)}=\\&\quad\frac{(c^*g_r+d^*h_r)\int_0^{\infty}h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^{\infty}h(t)\mathrm{d}t-1)},\end{aligned}$$

所以 $\|Au_n\|_S\leqslant\dfrac{(c^*g_r+d^*h_r)\int_0^{\infty}h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^{\infty}h(t)\mathrm{d}t-1)}$。

$$\begin{aligned}|D_{0+}^{\alpha-1}(Au_n)(t)|&\leqslant\int_0^t|f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))|\mathrm{d}s+\\&\quad\frac{1}{\int_0^{\infty}h(t)\mathrm{d}t-1}\int_0^{\infty}|f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))|\cdot|1-\int_s^{\infty}h(t)\mathrm{d}t|\mathrm{d}s\leqslant\\&\quad\frac{\int_0^{\infty}h(t)\mathrm{d}t}{\int_0^{\infty}h(t)\mathrm{d}t-1}\int_0^{\infty}|f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))|\mathrm{d}s\leqslant\\&\quad\frac{\int_0^{\infty}h(t)\mathrm{d}t\int_0^{\infty}(c(s)g_r+d(s)h_r)\mathrm{d}s}{\int_0^{\infty}h(t)\mathrm{d}t-1}=\\&\quad\frac{(c^*g_r+d^*h_r)\int_0^{\infty}h(t)\mathrm{d}t}{\int_0^{\infty}h(t)\mathrm{d}t-1},\end{aligned}$$

所以 $\|D_{0+}^{\alpha-1}(Au_n)\|_B\leqslant\dfrac{(c^*g_r+d^*h_r)\int_0^{\infty}h(t)\mathrm{d}t}{\int_0^{\infty}h(t)\mathrm{d}t-1}$，$\|Au_n\|\leqslant\dfrac{(c^*g_r+d^*h_r)\int_0^{\infty}h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^{\infty}h(t)\mathrm{d}t-1)}$，即$\{Au_n\}$有界。

定义函数

$$w_n(t)=\begin{cases}(Au_n)(t),\ \forall t\in J_k,\\(Au_n)(t_{k-1}^{+}),\ \forall t=t_{k-1}。\end{cases}$$

对 $\forall t_1,t_2\in J_k$,当 $t_2>t_1$ 时,有

$$\begin{aligned}&\left|\frac{(Au_n)(t_2)}{1+t_2^{\alpha-1}}-\frac{(Au_n)(t_1)}{1+t_1^{\alpha-1}}\right|=\\&\frac{1}{\Gamma(\alpha)(1+t_1^{\alpha-1})(1+t_2^{\alpha-1})}\left|\int_0^{t_1}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))\mathrm{d}s+\right.\\&\quad\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))\mathrm{d}s+\\&\quad\int_0^{t_1}((t_1t_2-t_1s)^{\alpha-1}-(t_1t_2-t_2s)^{\alpha-1})f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))\mathrm{d}s+\\&\quad\int_{t_1}^{t_2}(t_1t_2-t_1s)^{\alpha-1}f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))\mathrm{d}s+\\&\quad\left.\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\int_0^{\infty}h(t)\mathrm{d}t-1}\int_0^{\infty}f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))(1-\int_s^{\infty}h(t)\mathrm{d}t)\mathrm{d}s\right|\leqslant\\&\frac{1}{\Gamma(\alpha)}\left[\int_0^{t_1}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})\mid f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))\mid\mathrm{d}s+\right.\\&\quad\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mid f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))\mid\mathrm{d}s+\\&\quad\int_0^{t_1}((t_1t_2-t_1s)^{\alpha-1}-(t_1t_2-t_2s)^{\alpha-1})\mid f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))\mid\mathrm{d}s+\\&\quad\int_{t_1}^{t_2}(t_1t_2-t_1s)^{\alpha-1}\mid f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))\mid\mathrm{d}s+\\&\quad\left.\frac{\mid t_2^{\alpha-1}-t_1^{\alpha-1}\mid}{\int_0^{\infty}h(t)\mathrm{d}t-1}\int_0^{\infty}\mid f(s,u_n(s),D_{0+}^{\alpha-1}u_n(s))\mid\cdot\mid 1-\int_s^{\infty}h(t)\mathrm{d}t\mid\mathrm{d}s\right]\leqslant\\&\frac{1}{\Gamma(\alpha)}\left\{\int_0^{t_1}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})(c(s)g_r+d(s)h_r)\mathrm{d}s+\right.\\&\qquad(t_k-t_{k-1})^{\alpha-1}\int_{t_1}^{t_2}(c(s)g_r+d(s)h_r)\mathrm{d}s+\\&\qquad\int_0^{t_1}((t_1t_2-t_1s)^{\alpha-1}-(t_1t_2-t_2s)^{\alpha-1})(c(s)g_r+d(s)h_r)\mathrm{d}s+\\&\qquad\left.(t_k^2-t_{k-1}^2)^{\alpha-1}\int_{t_1}^{t_2}(c(s)g_r+d(s)h_r)\mathrm{d}s+\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\int_0^{\infty}h(t)\mathrm{d}t-1}\int_0^{\infty}(c(s)g_r+d(s)h_r)\mathrm{d}s\right\}。\end{aligned}$$

令 $g(t,s)=(t-s)^{\alpha-1}$,$g(t,s)\in C[\overline{J}_k\times\overline{J}_k,\mathbf{R}]$,由二元连续函数的一致连续性可知:存在 δ_1,使得 $\mid t_2-t_1\mid<\delta_1$ 时,$\mid g(t_2,s)-g(t_1,s)\mid=\mid(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\mid<\dfrac{\Gamma(\alpha)\varepsilon}{4(c^*g_r+d^*h_r)}$。

令 $g(t,r,s)=(tr-rs)^{\alpha-1}$,$g(t,r,s)\in C[\overline{J}_k\times\overline{J}_k\times\overline{J}_k,\mathbf{R}]$,由三元连续函数的一致连续性可知:存在 δ_2,使得 $\mid t_2-t_1\mid<\delta_2$ 时,$\mid g(t_2,t_1,s)-g(t_1,t_2,s)\mid=\mid(t_1t_2-t_1s)^{\alpha-1}-(t_1t_2-t_2s)^{\alpha-1}\mid<\dfrac{\Gamma(\alpha)\varepsilon}{4(c^*g_r+d^*h_r)}$。

由积分的绝对连续性可知:存在 δ_3,使得 $\mid t_2-t_1\mid<\delta_3$ 时,

$$\int_{t_1}^{t_2}[c(s)g_r(s)+d(s)h_r(s)]\mathrm{d}s<\frac{\Gamma(\alpha)\varepsilon}{4[(t_k-t_{k-1})^{\alpha-1}+(t_k^2-t_{k-1}^2)^{\alpha-1}]}。$$

由一元连续函数的一致连续性可知:存在 δ_4,使得 $\mid t_2-t_1\mid<\delta_4$ 时,

$$\mid t_2^{\alpha-1}-t_1^{\alpha-1}\mid<\frac{\Gamma(\alpha)(\int_0^{\infty}h(t)\mathrm{d}t-1)\varepsilon}{4(c^*g_r+d^*h_r)}。$$

取 $\delta=\min\{\delta_1,\delta_2,\delta_3,\delta_4\}$,当 $\mid t_2-t_1\mid<\delta$ 时,$\left|\dfrac{(Au_n)(t_2)}{1+t_2^{\alpha-1}}-\dfrac{(Au_n)(t_1)}{1+t_1^{\alpha-1}}\right|<\varepsilon$。所以 $w_n(t)$ 在 $\overline{J}_k$ 上是等度连续的。由 Ascoli-Arzela 定理可知,$w_n(t)$ 在 $\overline{J}_k$ 上有一致收敛的子列。因此,$\{(Au_n)(t)\}$ 在 J_k 上有一致收敛的子列。用对角线方法选取 $\{(Au_n)(t)\}$ 的子列 $\{(Au_{n_i})(t)\}(i=1,2,3,\cdots)$,使得 $\{(Au_{n_i})(t)\}$ 在每个 J_k 上

都一致收敛。设$\lim\limits_{i\to\infty}(Au_{n_i})(t)=w(t)$。

令

$$y_{n_i}(t)=\begin{cases}D_{0+}^{\alpha-1}(Au_{n_i})(t),\ \forall t\in J_k,\\ D_{0+}^{\alpha-1}(Au_{n_i})(t_{k-1}^{+}),\ \forall t=t_{k-1}。\end{cases}$$

$$\begin{aligned}&|D_{0+}^{\alpha-1}(Au_{n_i})(t_2)-D_{0+}^{\alpha-1}(Au_{n_i})(t_1)|=\\&\quad\left|\int_{t_1}^{t_2}f(s,u_{n_i}(s),D_{0+}^{\alpha-1}u_{n_i}(s))\mathrm{d}s\right|\leqslant\\&\quad\int_{t_1}^{t_2}|f(s,u_{n_i}(s),D_{0+}^{\alpha-1}u_{n_i}(s))|\mathrm{d}s\leqslant\\&\quad\int_{t_1}^{t_2}(c(s)g_r+d(s)h_r)\mathrm{d}s。\end{aligned}$$

由积分的绝对连续性可知:存在 δ_5,使得 $|t_2-t_1|<\delta_5$ 时,$\int_{t_1}^{t_2}c(s)g_r+d(s)h_r\mathrm{d}s<\varepsilon$。所以 $y_{n_i}(t)$ 在 $\overline{J}_k=[t_{k-1},t_k]$ 上是等度连续的。同样由 Ascoli-Arzela 定理可知,$\{y_{n_i}(t)\}$ 在 $\overline{J}_k$ 上有一致收敛的子列。因此,$\{D_{0+}^{\alpha-1}(Au_{n_i})(t)\}$ 在 J_k 上有一致收敛的子列。用对角线方法选取$\{D_{0+}^{\alpha-1}(Au_{n_i})(t)\}$的子列,使其在每个 J_k 上都一致收敛。为了使记号简便,假设$\{D_{0+}^{\alpha-1}(Au_{n_i}(t)\}$ 在每个 J_k 上都一致收敛。设

$$\lim_{i\to\infty}D_{0+}^{\alpha-1}(Au_{n_i})(t)=y(t)。$$

对 $\forall t\in J$,$Au_{n_i}(t)=I_{0+}^{\alpha-1}D_{0+}^{\alpha-1}(Au_{n_i})(t)=\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}D_{0+}^{\alpha-1}(Au_{n_i})(s)\mathrm{d}s$。因为 $D_{0+}^{\alpha-1}(Au_{n_i})(t)$ 有界,所以根据控制收敛定理可得:

$$\begin{aligned}&w(t)=\lim_{i\to\infty}Au_{n_i}(t)=\\&\quad\lim_{i\to\infty}\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}D_{0+}^{\alpha-1}(Au_{n_i})(s)\mathrm{d}s=\\&\quad\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\lim_{i\to\infty}D_{0+}^{\alpha-1}(Au_{n_i})(s)\mathrm{d}s=\\&\quad\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}y(s)\mathrm{d}s,\end{aligned}$$

所以 $D_{0+}^{\alpha-1}w(t)=y(t)$。因此,$w\in PC^1[J,\mathbf{R}]$。

因为 $c^*=\int_0^\infty c(t)\mathrm{d}t<\infty$,$d^*=\int_0^\infty d(t)\mathrm{d}t<\infty$,所以对 $\forall\varepsilon>0$,存在充分大的数 ξ,使得

$$\int_\xi^\infty(c(s)g_r+d(s)h_r)\mathrm{d}s<\frac{\varepsilon}{3}。$$

对 $\forall\xi<t<\infty$,

$$\begin{aligned}&|D_{0+}^{\alpha-1}(Au_{n_i})(t)-D_{0+}^{\alpha-1}(Au_{n_i})(\xi)|=\left|\int_\xi^t f(s,u_{n_i}(s),D_{0+}^{\alpha-1}u_{n_i}(s))\mathrm{d}s\right|\leqslant\\&\quad\int_\xi^t|f(s,u_{n_i}(s),D_{0+}^{\alpha-1}u_{n_i}(s))|\mathrm{d}s\leqslant\int_\xi^t(c(s)g_r+d(s)h_r)\mathrm{d}s\leqslant\\&\quad\int_\xi^\infty(c(s)g_r+d(s)h_r)\mathrm{d}s<\frac{\varepsilon}{3},\end{aligned}$$

因为$\lim\limits_{i\to\infty}D_{0+}^{\alpha-1}(Au_{n_i})(t)=D_{0+}^{\alpha-1}w(t)$,所以令 $i\to\infty$,可得 $|D_{0+}^{\alpha-1}w(t)-D_{0+}^{\alpha-1}w(\xi)|<\frac{\varepsilon}{3}$。

在$[0,\xi]$ 上,当 $i\to\infty$ 时,$D_{0+}^{\alpha-1}(Au_{n_i})(t)$ 一致收敛于 $D_{0+}^{\alpha-1}w(t)$,所以存在 i_0,当 $i>i_0$ 时,$|D_{0+}^{\alpha-1}(Au_{n_i})(t)-D_{0+}^{\alpha-1}w(t)|<\frac{\varepsilon}{3}$。

对 $\forall\xi<t<\infty$,当 $i>i_0$ 时,

$$\begin{aligned}&|D_{0+}^{\alpha-1}(Au_{n_i})(t)-D_{0+}^{\alpha-1}w(t)|\leqslant\\&|D_{0+}^{\alpha-1}(Au_{n_i})(t)-D_{0+}^{\alpha-1}(Au_{n_i})(\xi)|+|D_{0+}^{\alpha-1}(Au_{n_i})(\xi)-D_{0+}^{\alpha-1}w(\xi)|+|D_{0+}^{\alpha-1}w(\xi)-D_{0+}^{\alpha-1}w(t)|<\varepsilon。\end{aligned}$$

所以对 $\forall t\in J$,当 $i\to\infty$ 时,有 $|D_{0+}^{\alpha-1}(Au_{n_i})(t)-D_{0+}^{\alpha-1}w(t)|<\varepsilon$,所以 $\|D_{0+}^{\alpha-1}Au_{n_i}-D_{0+}^{\alpha-1}w\|_B<\varepsilon$。因此,$\lim\limits_{i\to\infty}\|D_{0+}^{\alpha-1}Au_{n_i}-D_{0+}^{\alpha-1}w\|_B=0$。

对 $\forall t\in J$,$(Au_{n_i})(t)=I_{0+}^{\alpha-1}(D_{0+}^{\alpha-1}(Au_{n_i})(t))$,$w(t)=I_{0+}^{\alpha-1}(D_{0+}^{\alpha-1}w(t))$。所以

$$\frac{|(Au_{n_i})(t)-w(t)|}{1+t^{\alpha-1}}=\frac{1}{\Gamma(\alpha-1)(1+t^{\alpha-1})}\left|\int_0^t(t-s)^{\alpha-2}(D_{0+}^{\alpha-1}(Au_{n_i})(s)-D_{0+}^{\alpha-1}w(s))\mathrm{d}s\right|\leqslant$$

$$\frac{1}{\Gamma(\alpha-1)(1+t^{\alpha-1})}\int_0^t (t-s)^{\alpha-2}\mid D_{0+}^{\alpha-1}(Au_{n_i})(s)-D_{0+}^{\alpha-1}w(s)\mid \mathrm{d}s\leqslant$$
$$\frac{1}{\Gamma(\alpha-1)(1+t^{\alpha-1})}\parallel D_{0+}^{\alpha-1}(Au_{n_i})-D_{0+}^{\alpha-1}w\parallel_B\int_0^t (t-s)^{\alpha-2}\mathrm{d}s\leqslant$$
$$\frac{1}{\Gamma(\alpha)}\parallel D_{0+}^{\alpha-1}(Au_{n_i})-D_{0+}^{\alpha-1}w\parallel_B,$$

所以$\lim\limits_{i\to\infty}\parallel Au_{n_i}-w\parallel_S=0$。因此,$\lim\limits_{i\to\infty}\parallel Au_{n_i}-w\parallel=0$,即 A 是紧的。因此,A 是全连续算子。根据定理 2 可知边值问题(1) 在 J 上至少有 1 个解。

4 举 例

例 1 考虑半无穷区间上分数阶脉冲微分方程边值问题

$$\begin{cases} D_{0+}^{\frac{3}{2}}u(t)=\mathrm{e}^{-9t}\left[\dfrac{\mid u(t)\mid}{1+t^{\frac{1}{2}}+\mid u(t)\mid}+\dfrac{\mid D_{0+}^{\frac{1}{2}}u(t)\mid}{1+\mid D_{0+}^{\frac{1}{2}}u(t)\mid}\right], \\ \Delta D_{0+}^{\frac{1}{2}}u\mid_{t=k}=\dfrac{\parallel u\parallel}{2^{k+6}(1+\parallel u\parallel)},\quad k=1,2,\cdots, \\ u(0)=0,\quad D_{0+}^{\frac{1}{2}}u(\infty)=\displaystyle\int_0^\infty h(t)D_{0+}^{\frac{1}{2}}u(t)\mathrm{d}t \end{cases}$$

解的存在性,其中$\int_0^\infty h(t)\mathrm{d}t=\frac{3}{2}$。

解 $\mid f(t,u(t)D_{0+}^{\alpha-1}u(t))-f(t,v(t),D_{0+}^{\alpha-1}v(t))\mid\leqslant \mathrm{e}^{-9t}\dfrac{\mid u(t)-v(t)\mid}{1+t^{\frac{1}{2}}}+\mathrm{e}^{-9t}\mid D_{0+}^{\frac{1}{2}}u(t)-D_{0+}^{\frac{1}{2}}v(t)\mid$,

$$\mid I_k(u(t_k^-))-I_k(v(t_k^-))\mid\leqslant\frac{\mid\parallel u\parallel-\parallel v\parallel\mid}{2^{k+6}}\leqslant\frac{\parallel u-v\parallel}{2^{k+6}},$$

所以,$a(t)=b(t)=\mathrm{e}^{-9t}$,$\gamma_k=\dfrac{1}{2^{k+6}}$。$a^*=b^*=\dfrac{1}{9}<1$,$\gamma^*=\dfrac{1}{64}<1$。$\dfrac{7}{8}<\dfrac{1}{2}\Gamma(\dfrac{1}{2})=\Gamma(\dfrac{3}{2})=\dfrac{\sqrt{\pi}}{2}<1$,

$\rho=\dfrac{\frac{3}{2}\times\left(\frac{2}{9}+\frac{1}{64}\right)}{\Gamma(\frac{3}{2})\times(\frac{3}{2}-1)}<\dfrac{137}{192}\times\dfrac{8}{7}=\dfrac{137}{168}<1$,满足定理 3 的条件。因此,该分数阶脉冲微分方程边值问题有且只有 1 个解。

例 2 考虑半无穷区间上分数阶脉冲微分方程边值问题

$$\begin{cases} D_{0+}^{\frac{3}{2}}u(t)=\mathrm{e}^{-4t}\left(\dfrac{u(t)}{1+t^{\frac{1}{2}}}\right)^3+\mathrm{e}^{-6t}(D_{0+}^{\frac{1}{2}}u(t))^2, \\ \Delta D_{0+}^{\frac{1}{2}}u\mid_{t=k}=k^{-1}3^{-k-4}\parallel u\parallel,\quad k=1,2,\cdots, \\ u(0)=0,D_{0+}^{\frac{1}{2}}u(\infty)=\displaystyle\int_0^\infty h(t)D_{0+}^{\frac{1}{2}}u(t)\mathrm{d}t \end{cases}$$

解的存在性,其中$\int_0^\infty h(t)\mathrm{d}t=\frac{3}{2}$。

解 $\mid I_k(u(t_k^-))-I_k(v(t_k^-))\mid=\mid k^{-1}3^{-k-4}(\parallel u\parallel-\parallel v\parallel)\mid\leqslant 3^{-k-4}\parallel u-v\parallel$,$\gamma_k=3^{-k-4}$,$\gamma^*=\dfrac{1}{162}$,所以条件 H_2) 成立。且满足条件:

$$\frac{\gamma^*\int_0^\infty h(t)\mathrm{d}t}{\Gamma(\alpha)(\int_0^\infty h(t)\mathrm{d}t-1)}=\frac{\frac{1}{162}\times\frac{3}{2}}{\frac{1}{2}\Gamma(\frac{3}{2})}<\frac{1}{54}\times\frac{8}{7}=\frac{4}{189}<1,$$

$$\mid f(t,u(t),D_{0+}^{\alpha-1}u(t))\mid\leqslant \mathrm{e}^{-4t}\left|\frac{u(t)}{1+t^{\frac{1}{2}}}\right|^3+\mathrm{e}^{-6t}\mid D_{0+}^{\frac{1}{2}}u(t)\mid^2,$$

所以 $c(t)=\mathrm{e}^{-4t}$,$d(t)=\mathrm{e}^{-6t}$,$c^*=\dfrac{1}{4}<\infty$,$d^*=\dfrac{1}{6}<\infty$,$g\left(\dfrac{u}{1+t^{\alpha-1}}\right)=\left(\dfrac{\mid u(t)\mid}{1+t^{\frac{1}{2}}}\right)^3$,$h(\tilde{u})=\mid D_{0+}^{\frac{1}{2}}u(t)\mid^2$,所以条件 H_3) 成立。

$\mid I_k(u(t_k^-))\mid=k^{-1}3^{-k-4}\parallel u\parallel\leqslant 3^{-k-4}\parallel u\parallel$,所以 $\eta_k=3^{-k-4}$,$F(u)=\parallel u\parallel$,$\eta^*=\dfrac{1}{162}<\infty$,所以条件

H_4）成立。

因此，根据定理 4 可得该分数阶脉冲微分方程边值问题至少有 1 个解。

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